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Existence of solution a fractional differential equation

Zouaoui Bekri[1,*] and Slimane Benaicha[2]

[1] Laboratory of fundamental and applied mathematics, University of Oran 1, Ahmed Ben Bella, Es-senia, 31000 Oran, Algeria.
[2] Laboratory of fundamental and applied mathematics, University of Oran 1, Ahmed Ben Bella, Es-senia, 31000 Oran, Algeria.
* Correspondence: zouaouibekri@yahoo.fr

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Abstract: In this paper, we study the existence of nontrivial solution for the fractional differential equation of order α with three point boundary conditions having the following form

$$D^{\alpha}u(t) = f(t, v(t), D^{\nu}v(t)), \quad t \in (0, T)$$

$$u(0) = 0, \quad u(T) = au(\xi),$$

where $1 < \alpha < 2$, $\nu, a > 0$, $\xi \in (0, T)$, $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$. D is the standard Riemann-Liouville fractional derivative operator and $f \in C([0, 1] \times \mathbf{R}^2, \mathbf{R})$. Applying the Leray-Schauder nonlinear alternative we prove the existence of at least one solution. As an application, we also given some examples to illustrate the results obtained.

Keywords: Caputo derivative of fractional order, Leary-Schauder nonlinear alternative, fixed point theorem, Riemann-Liouville fractional integral, fractional differential equations.

MSC: 34A08, 34A34, 36A33.

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1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, signal and image processing, capacitor theory, electrical circuits, electron-analytical chemistry, biology, ow in porous media, aerodynamics, viscoelasticity, control theory, fitting of experimental data, and so forth, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes (for details, see [1–9]). The fractional differential equations under various conditions have been studied by ([10–13]), etc. The three point boundary value problem given by a coupled system of FDE on the interval $(0, 1)$ was studied by Bashir [10]

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), & t \in (0, 1), \\ D^{\beta}v(t) = f(t, u(t), D^{q}u(t)), & t \in (0, 1), \\ u(0) = 0, \;\; u(1) = au(\xi), \;\; v(0) = 0, \;\; v(1) = av(\xi), & \end{cases} \tag{1}$$

where $1 < \alpha, \beta < 2$, $p, q, a > 0$, $0 < \xi < 1$, $\alpha - q \geq 1$, $\beta - p \geq 1$, $a\xi^{\alpha-1} < 1$ and $a\xi^{\beta-1} < 1$. D is the standard Riemann-Liouville fractional derivative operator and $f : [0, 1] \times \mathbf{R}^2 \longrightarrow \mathbf{R}^2$.

Infinite systems of ODE's was first studied by Persidskii [14] with the aid of classical tools such as successive approximation and the classical Banach fixed point principle. The infinite systems of differential equations emerge in study of various topics of nonlinear analysis. For example semidiscretization of certain parabolic partial differential equation leads to an infinite system of ODE [15], while modeling certain physical phenomenon in theory of neural sets, branching process and mechanics ([16,17]), where the infinite system can be represented as an ordinary differential equation. Consider the following infinite system of fractional differential equations [18]

$$\begin{cases} D^{\alpha}u_i(t) = f_i(t, u(t)), & t \in (0, T) \\ u_i(0) = u_i^0 = 0, \quad u_i(T) = au_i(\xi), & i = 1, 2, 3... \\ 1 < \alpha < 2, \quad a\xi^{\alpha-1} < T^{\alpha-1}, \end{cases} \tag{2}$$

where each $u_i(t)$ is a differentiable function of class $C^{[\alpha]+1}$. We will denote the sequence $\{u_i(t)\}_{i=1}^{\infty} = u(t)$, $\{u_i(0)\}_{i=1}^{\infty} = u_0$, $\{u_i(\xi)\}_{i=1}^{\infty} = u(\xi)$ and $\{f_i(t, u(t))\}_{i=1}^{\infty} = f(t, u(t))$ which is an element of some Banach sequence space $(E, \|.\|)$.

Motivated by the above works, the aim of this paper is to establish some sufficient conditions for the existence of nontrivial solution for the fractional differential equations (FDE) as follows

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{\nu}v(t)), & t \in (0, T) \\ u(0) = 0, \quad u(T) = au(\xi), \end{cases} \tag{3}$$

where $1 < \alpha < 2$; $\nu, a > 0$, $\xi \in (0, T)$; $\alpha - \mu \geq 1$ and $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$. D is the standard Riemann-Liouville fractional derivative operator and $f \in C([0, 1] \times \mathbf{R}^2, \mathbf{R})$.

This paper is organized as follows. In Section 2, we present some definitions and lemmas that will be used to prove the results. Then, in Section 3, we present and prove our main results which consists of existence theorems and corollary for nontrivial solution of the FDE 3, and we establish some existence criteria of at least one solution by using the Leray-Schauder nonlinear alternative. Finally, in Section 4, as an application, we give some examples to illustrate the results we obtained.

2. Preliminaries

In this section, we introduce some necessary definitions and lemmas of fractional calculus to facilitate the analysis of the Problem (3). These details can be found in the recent literature, see ([3,7,19–23]) and the references therein.

Definition 1. Let $\alpha > 0$, $n - 1 < \alpha < n$, $n = [\alpha] + 1$ and $u \in C([0, 1), \mathbf{R})$. The Caputo derivative of fractional order α for the function u is defined by

$$^cD^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \longrightarrow \mathbf{R}$ is given by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \ \ t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 1. *([23]) Let $\alpha, \beta > 0$ and $u \in L^p(0, 1)$, $1 \leq p \leq +\infty$. Then the next formulas hold;*
(i) $(I^{\beta}I^{\alpha}u)(t) = I^{\alpha+\beta}u(t)$,
(ii) $(D^{\beta}I^{\alpha}u)(t) = I^{\alpha-\beta}u(t)$,
(iii) $(D^{\alpha}I^{\alpha}u)(t) = u(t)$.

Lemma 2. *Let $\alpha > 0$, $n - 1 < \alpha < n$ and the function $g : [0; T] \longrightarrow \mathbf{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^cD^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Lemma 3. *Assume that $u \in C[0,1] \cap L^1(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $u \in C^n[0,1]$, then*

$$I^{\alpha\; c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Lemma 4. *For $\alpha > 0$, the general solution of the fractional differential equation $D^{\alpha}u(t) = 0$ with $u \in C[0,1] \cap L^1(0,1)$ is given by*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

where $c_i \in \mathbf{R}$, $i = 1, 2, ..., n$. Hence for $u \in C[0,1] \cap L^1(0,1)$, we have

$$I^{\alpha}D^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}.$$

Lemma 5. *Let $y \in C([0,T])$, $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$. Then FDE*

$$\begin{cases} D^{\alpha}u(t) = y(t), & t \in (0,T) \\ u(0) = 0,\ u(T) = au(\xi), \end{cases}$$

has a unique solution

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \left[(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1} - a\xi^{\alpha-1})}\right] y(s)ds - \frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)}\int_t^T (t(T-s))^{\alpha-1}y(s)ds \\ &\quad + \frac{a}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(t(\xi-s))^{\alpha-1}y(s)ds. \end{aligned}$$

Proof. ([10])The general solution of FDE is

$$u(t) = I^{\alpha}y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},\ where\ \ c_1, c_2 \in \mathbf{R}.$$

Using the boundary conditions, we find that $c_2 = 0$ and

$$c_1 = -\frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})}\left[\int_0^T \frac{y(s)ds}{(T-s)^{\alpha-1}\Gamma(\alpha)} - a\int_0^{\xi}\frac{y(s)ds}{(\xi-s)^{\alpha-1}\Gamma(\alpha)}\right].$$

Substituting c_1 and c_2 by their values in $u(t)$, we obtain the solution in the statement of the lemma. This completes the proof. □

Define the integral operator $F : E \to E$, by

$$\begin{aligned} Fu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \left[(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1} - a\xi^{\alpha-1})}\right] f(s, v(s), D^{\nu}v(s))ds \\ &\quad - \frac{1}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)}\int_t^T (t(T-s))^{\alpha-1} f(s, v(s), D^{\nu}v(s))ds \\ &\quad + \frac{a}{(T^{\alpha-1} - a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(t(\xi-s))^{\alpha-1} f(s, v(s), D^{\nu}v(s))ds. \end{aligned}$$

By Lemma 5, the FDE (3) has a solution if and only if the operator F has a fixed point in E. So we only need to seek a fixed point of F in E. By Ascoli-Arzela theorem, we can prove that F is a completely continuous operator. Now we cite the Leray-Schauder nonlinear alternative.

Lemma 6. *Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$. $F : \overline{\Omega} \to E$ be a completely continuous operator. Then, either*

(i) there exists $u \in \partial\Omega$ and $\lambda > 1$ such that $F(u) = \lambda u$, or

(ii) there exists a fixed point $u^* \in \overline{\Omega}$ of F.

3. Main results

In this section, we prove the existence of a nontrivial solution for the FDE (3). Let $E = C([0,T])$ with the norm $\|v\| = \max_{t\in[0,T]}\{|v(t)|, |D^\nu v(t)|\}$ for any $v \in E$, $f \in C([0,T]\times \mathbf{R}^2, \mathbf{R})$.

Theorem 1. *Suppose that $f(t,0,0) \neq 0$, $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$, and there exist nonnegative functions $k, h, l \in L^1[0,T]$ such that*

$$|f(t,x,y)| \leq k(t)|x| + h(t)|y| + l(t), \quad a.e.\ (t,x,y) \in [0,T]\times \mathbf{R}^2,$$

and

$$\frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(k(s)+h(s))ds + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}(k(s)+h(s))ds < 1.$$

Then the FDE (3) has at least one nontrivial solution $u^* \in C([0,T])$.

Proof. Let

$$A = \frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(k(s)+h(s))ds + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}(k(s)+h(s))ds,$$

and

$$B = \frac{2T^{\alpha-1} + a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}l(s)ds + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}l(s)ds,$$

then $A < 1$. Since $f(t,0,0) \neq 0$, there exists an interval $[a,b] \subset [0,1]$ such that $\min_{a\leq t\leq b}|f(t,0,0)| > 0$, and, as $l(t) \geq |f(t,0,0)|$, a.e., and $t \in [0,T]$, so $B > 0$.

Let $C = B(1-A)^{-1}$ and $\Omega = \{(u,v) \in E^2 : \|(u,v)\|_{E^2} < C\}$. Assume that $u \in \partial\Omega$ and $\lambda > 1$ such that $Fu = \lambda u$, then

$$\begin{aligned}
\lambda C &= \lambda\|u\| = \|Fu\| = \max_{0\leq t\leq T}|(Fu)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,T]}\int_0^t \left|(t-s)^{\alpha-1} - \frac{(t(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right| |f(s,v(s),D^\nu v(s))|ds \\
&\quad + \max_{t\in[0,T]}\frac{1}{|T^{\alpha-1}-a\xi^{\alpha-1}|\Gamma(\alpha)}\int_t^T (t(T-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds \\
&\quad + \max_{t\in[0,T]}\frac{a}{|T^{\alpha-1}-a\xi^{\alpha-1}|\Gamma(\alpha)}\int_0^\xi (t(\xi-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^T \left[(T-s)^{\alpha-1} + \frac{(T(T-s))^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})}\right]|f(s,v(s),D^\nu v(s))|ds \\
&\quad + \frac{a}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (T(\xi-s))^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds \\
&\leq \frac{(2T^{\alpha-1}+a\xi^{\alpha-1})}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(k(s)|v(s)| + h(s)|D^\nu v(s)| + l(s))ds \\
&\quad + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}(k(s)|v(s)| + h(s)|D^\nu v(s)| + l(s))ds \\
&\leq \left[\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(k(s)+h(s))\|v\|ds\right. \\
&\quad \left. + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}(k(s)+h(s))\|v\|ds\right] \\
&\quad + \left[\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}l(s)ds + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}l(s)ds\right] \\
&= A\|v\| + B.
\end{aligned}$$

Therefore, $\lambda \leq A + \frac{B}{C} = A + \frac{B}{B(1-A)^{-1}} = A + (1-A) = 1$. This contradicts $\lambda > 1$. By Lemma 6, F has a fixed point $u^* \in \overline{\Omega}$. In view of $f(t,0,0) \neq 0$, the FDE (3) has a nontrivial solution $u^* \in E$.

Now, we prove that the operator F is completely continuous, we have $B_C = \{v \in E : ||v|| \leq C\}$ is a bounded closed convex set of E. We shall prove that $F(B_C)$ is relatively compact. The proof will be done is some steps.

(i) Let $v \in B_C$, we have $|Fu(t)| \leq A\|v\| + B$. Consequently $F(B_C)$ is uniformly bounded.
(ii) Let us prove that $F(B_C)$ is equicontinuous. Let $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, and $v \in B_C$, we have

$$\begin{aligned}
\Big|Fu(t_1) - Fu(t_2)\Big| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - \frac{(t_1(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right] f(s,v(s),D^\nu v(s))ds\\
&\quad - \frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{T}(t_1(T-s))^{\alpha-1} f(s,v(s),D^\nu v(s))ds\\
&\quad + \frac{a}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(t_1(\xi-s))^{\alpha-1} f(s,v(s),D^\nu v(s))ds\\
&\quad - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-s)^{\alpha-1} - \frac{(t_2(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right] f(s,v(s),D^\nu v(s))ds\\
&\quad + \frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_2}^{T}(t_2(T-s))^{\alpha-1} f(s,v(s),D^\nu v(s))ds\\
&\quad - \frac{a}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(t_2(\xi-s))^{\alpha-1} f(s,v(s),D^\nu v(s))ds\Big|\\
&= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - \frac{(t_1(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right] f(s,v(s),D^\nu v(s))ds\\
&\quad - \frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{T}(t_1(T-s))^{\alpha-1} f(s,v(s),D^\nu v(s))ds\\
&\quad - \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-s)^{\alpha-1} - \frac{(t_2(T-s))^{\alpha-1}}{(T^{\alpha-1}-a\xi^{\alpha-1})}\right] f(s,v(s),D^\nu v(s))ds\\
&\quad + \frac{1}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_2}^{T}(t_2(T-s))^{\alpha-1} f(s,v(s),D^\nu v(s))ds\\
&\quad + \frac{a(t_1^{\alpha-1}-t_2^{\alpha-1})}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1} f(s,v(s),D^\nu v(s))ds\Big|\\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1} + \frac{(t_1(T-s))^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})}\right] |f(s,v(s),D^\nu v(s))|ds\\
&\quad + \frac{1}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{T}(t_1(T-s))^{\alpha-1} |f(s,v(s),D^\nu v(s))|ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_2}\left[(t_2-s)^{\alpha-1} + \frac{(t_2(T-s))^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})}\right] |f(s,v(s),D^\nu v(s))|ds\\
&\quad + \frac{1}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_2}^{T}(t_2(T-s))^{\alpha-1} |f(s,v(s),D^\nu v(s))|ds\\
&\quad + \frac{a|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1} |f(s,v(s),D^\nu v(s))|ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Big|Fu(t_1) - Fu(t_2)\Big| &\leq \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t_1}\left[((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}) + \frac{[(t_1(T-s))^{\alpha-1} - (t_2(T-s))^{\alpha-1}]}{(T^{\alpha-1}+a\xi^{\alpha-1})}\right]\\
&\times |f(s,v(s),D^\nu v(s))|ds + \frac{1}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_{t_1}^{t_2}[(t_1(T-s))^{\alpha-1} - (t_2(T-s))^{\alpha-1}]\\
&\times |f(s,v(s),D^\nu v(s))|ds + \frac{a|t_1^{\alpha-1}-t_2^{\alpha-1}|}{(T^{\alpha-1}-a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}|f(s,v(s),D^\nu v(s))|ds.
\end{aligned}$$

Letting $t_1 \to t_2$, then $|Fu(t_1) - Fu(t_2)|$ tends to 0. Consequently $F(B_C)$ is equicontinuous. From Ascoli-Arzela theorem, we deduce that F is a completely continuous. This completes the proof.

□

Theorem 2. *Suppose that $f(t,0,0) \neq 0$, $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$, and there exist nonnegative functions $k,h,l \in L^1[0,T]$ such that $|f(t,x,y)| \leq k(t)|x| + h(t)|y| + l(t)$, a.e. $(t,x,y) \in [0,T] \times \mathbf{R}^2$. If one of the following conditions is fulfilled;*

(1) There exists a constant $p > 1$ such that

$$\int_0^1 (k(s)+h(s))^p ds < \left[\frac{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q} + aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}\right]^p, \ \frac{1}{p}+\frac{1}{q} = 1,$$

(2) $k(s)+h(s)$ satisfies

$$k(s)+h(s) \leq \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^\alpha + aT^{\alpha-1}\xi^\alpha}, \quad a.e. \quad s \in [0,T],$$

$$meas\left\{s \in [0,T] : k(s)+h(s) < \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^\alpha + aT^{\alpha-1}\xi^\alpha}\right\} > 0.$$

Then the FDE (3) has at least one nontrivial solution $u^* \in E$.

Proof. Let A be defined as in the proof of Theorem 1. To prove Theorem 2, we only need to prove that $A < 1$. Since $T^{\alpha-1} + a\xi^{\alpha-1} \neq 0$, we have

$$A = \frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(k(s)+h(s))ds + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1}(k(s)+h(s))ds.$$

(1) Using the Hölder inequality, we have

$$\begin{aligned}
A &\leq \left[\int_0^1 (k(s)+h(s))^p ds\right]^{1/p} \left\{\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \left[\int_0^T \left((T-s)^{\alpha-1}\right)^q ds\right]^{1/q}\right.\\
&\quad \left. + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \left[\int_0^\xi \left((\xi-s)^{\alpha-1}\right)^q ds\right]^{1/q}\right\}\\
&\leq \left[\int_0^1 (k(s)+h(s))^p ds\right]^{1/p} \left\{\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \left[\frac{T^{1+q(\alpha-1)}}{(1+q(\alpha-1))}\right]^{1/q}\right.\\
&\quad \left. + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \left[\frac{\xi^{1+q(\alpha-1)}}{1+q(\alpha-1)}\right]^{1/q}\right\}\\
&\leq \left[\int_0^1 (k(s)+h(s))^p ds\right]^{1/p} \left[\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}} + \frac{aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}\right]\\
&\leq \frac{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q} + aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}} \times \frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q} + aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}\\
&= 1.
\end{aligned}$$

(2) In this case, we have

$$\begin{aligned}
A \leq \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^\alpha + aT^{\alpha-1}\xi^\alpha} &\left[\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds\right.\\
&\left. + \frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1} ds\right]
\end{aligned}$$

$$\leq \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}\left[\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}}{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}+\frac{aT^{\alpha-1}\xi^{\alpha}}{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}\right]$$
$$\leq \frac{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}.\frac{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{\alpha}+aT^{\alpha-1}\xi^{\alpha}}{\alpha\Gamma(\alpha)(T^{\alpha-1}+a\xi^{\alpha-1})}=1.$$

This completes the proof. □

Corollary 1. *Suppose $f(t,0,0)\neq 0$, $(1+a)T^{\alpha-1}\neq 0$, and there exist nonnegative functions $k,h,l\in L^1[0,T]$ such that $|f(t,x,y)|\leq k(t)|x|+h(t)|y|+l(t)$, a.e. $(t,x,y)\in[0,T]\times\mathbf{R}^2$. If one of following conditions is fulfilled;*

(1) There exists a constant $p>1$ such that

$$\int_0^1 (k(s)+h(s))^p ds < \left[\frac{(1+a)T^{\alpha-1}\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{2(1+a)T^{\alpha-1}T^{(1+q(\alpha-1))/q}}\right]^p,\ \frac{1}{p}+\frac{1}{q}=1.$$

(2) $k(s)+h(s)$ satisfies

$$k(s)+h(s)\leq\frac{\alpha\Gamma(\alpha)}{2T^{\alpha}},\quad a.e.\ \ s\in[0,T],$$
$$meas\left\{s\in[0,T]:k(s)+h(s)<\frac{\alpha\Gamma(\alpha)}{2T^{\alpha}}\right\}>0.$$

Then, the FDE (3) has at least one nontrivial solution $u^*\in E$.

Proof. In this case, we have

$$A=\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^{\xi}(\xi-s)^{\alpha-1}(k(s)+h(s))ds$$
$$\leq\frac{2T^{\alpha-1}+aT^{\alpha-1}}{(T^{\alpha-1}+aT^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+aT^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds$$
$$=\frac{2(1+a)T^{\alpha-1}}{(1+a)T^{\alpha-1}\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds.$$

Proof of this Corollary 1 is similar to the proof Theorem 2. This completes the proof. □

4. Applications

In order to illustrate the above results, we consider some examples.

Example 1. Consider the following system of FDE

$$\begin{cases} D^{3/2}u(t)=\frac{t}{207}v(t)+\frac{t+2}{100}D^{5/4}v(t)+t^2-1, & t\in(0,T)\\ u(0)=0,\quad u(T)=2u(T/2). \end{cases}\tag{4}$$

Set $\alpha=3/2$, $a=2$, $\xi=T/2$, and

$$f(t,x,y)=\frac{t}{207}x(t)+\frac{t+2}{100}y(t)+t^2-1,$$
$$k(t)=\frac{t}{100},\quad h(t)=\frac{t+2}{100},\quad l(t)=t^2.$$

It is easy to prove that $k,h,l\in L^1[0,T]$ are nonnegative functions, and

$$|f(t,x,y)|\leq k(t)|x|+h(t)|y|+l(t),\quad a.e.\ \ (t,x,y)\in[0,T]\times\mathbf{R}^2,$$

and

$$T^{\alpha-1}+a\xi^{\alpha-1}=(1+\frac{2}{2^{1/2}})T^{1/2}\neq 0.$$

Moreover, we have

$$A=\frac{2T^{\alpha-1}+a\xi^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}(k(s)+h(s))ds+\frac{aT^{\alpha-1}}{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)}\int_0^\xi(\xi-s)^{\alpha-1}(k(s)+h(s))ds,$$

$$A\approx 13.10^{-3}.T^{3/2}+4.10^{-3}.T^{5/2}<1.$$

Hence, by Theorem 1, the FDE (4) has at least one nontrivial solution u^* in E.

Example 2. Consider the following system of FDE

$$\begin{cases} D^{1/2}u(t)=\frac{\sqrt[3]{1+t^5}}{20}v(t)\sin v(t)+\frac{\sqrt[3]{1+t^5}}{5}D^{3/4}v(t)+\cos t-e^t, \quad t\in(0,T) \\ u(0)=0, \quad u(T)=4u(T/3). \end{cases} \tag{5}$$

Set $\alpha=1/2$, $a=4$, $\xi=T/3$, and

$$f(t,x,y)=\frac{\sqrt[3]{1+t^5}}{20}x(t)\sin x(t)+\frac{\sqrt[3]{1+t^5}}{5}y(t)+\cos t-e^t,$$

$$k(t)=\sqrt[3]{1+t^5}/10, \quad h(t)=\sqrt[3]{1+t^5}/4, \quad l(t)=\cos t+e^t.$$

It is easy to prove that $k,h,l\in L^1[0,T]$ are nonnegative functions, and

$$|f(t,x,y)|\le k(t)|x|+h(t)|y|+l(t), \quad a.e. \ (t,x,y)\in[0,T]\times\mathbf{R}^2,$$

and

$$T^{\alpha-1}+a\xi^{\alpha-1}=(1+\frac{4}{3^{-1/2}})T^{-1/2}\neq 0.$$

Let $p=3,\ q=3/2$, such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_0^1(k(s)+h(s))^p ds=\frac{2401}{48000}\approx 0.05.$$

Moreover, we have

$$\left[\frac{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}+aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}\right]^p\approx 0.51.T^{-1/2}.$$

Therefore,

$$\int_0^1(k(s)+h(s))^p ds<\left[\frac{(T^{\alpha-1}+a\xi^{\alpha-1})\Gamma(\alpha)(1+q(\alpha-1))^{1/q}}{(2T^{\alpha-1}+a\xi^{\alpha-1})T^{(1+q(\alpha-1))/q}+aT^{\alpha-1}\xi^{(1+q(\alpha-1))/q}}\right]^p.$$

Hence, by Theorem 2(1), the FDE (5) has at least one nontrivial solution u^* in E.

Example 3. Consider the following system of FDE

$$\begin{cases} D^{3/2}u(t)=\frac{\sqrt{t}}{2(\frac{1}{2}+v(t))}e^{|v^2(t)-1|}\cos v(t)+\frac{(1+t^2)}{9+e^t}D^{7/3}v(t)+e^{-t}-\sin t, \quad t\in(0,T) \\ u(0)=0, \quad u(T)=3u(T/4). \end{cases} \tag{6}$$

Set $\alpha=3/2$, $a=3$, $\xi=T/4$, and

$$f(t,x,y)=\frac{\sqrt{t}}{2(\frac{1}{2}+x(t))}e^{|x^2(t)-1|}\cos x(t)+\frac{(1+t^2)}{9+e^t}y(t)+e^{-t}-\sin t,$$

$$k(t) = \frac{\sqrt{t}}{2}, \quad h(t) = \frac{(1+t^2)}{3}, \quad l(t) = e^{-t} + \sin t.$$

It is easy to prove that $k, h, l \in L^1[0, T]$ are nonnegative functions, and

$$|f(t,x,y)| \le k(t)|x| + h(t)|y| + l(t), \quad a.e. \ (t,x) \in [0,T] \times \mathbf{R}^2,$$

and

$$T^{\alpha-1} + a\xi^{\alpha-1} = (1 + \frac{3}{4^{1/2}})T^{1/2} \neq 0.$$

Moreover, we have

$$\frac{\alpha\Gamma(\alpha)(T^{\alpha-1} + a\xi^{\alpha-1})}{(2T^{\alpha-1} + a\xi^{\alpha-1})T^{\alpha} + aT^{\alpha-1}\xi^{\alpha}} = \frac{15\sqrt{\pi}}{31}T^{-3/2}.$$

Therefore,

$$k(s) + h(s) = \frac{\sqrt{s}}{2} + \frac{(1+s^2)}{3} < \frac{15\sqrt{\pi}}{31}T^{-3/2}, \quad s \in [0,T],$$

$$meas\{s \in [0,T] : k(s) + h(s) < \frac{\alpha\Gamma(\alpha)(T^{\alpha-1} + a\xi^{\alpha-1})}{(2T^{\alpha-1} + a\xi^{\alpha-1})T^{\alpha} + aT^{\alpha-1}\xi^{\alpha}}\} > 0.$$

Hence, by Theorem 2(2), the FDE (6) has at least one nontrivial solution u^* in E.

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