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*Article*

# Higer-order commutators of parametrized Marcinkewicz integrals on Herz spaces with variable exponent

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**Abstract:** Let $0 < \rho < n$ and $\mu_\Omega^\rho$ be the Parametrized Marcinkiewicz integrals operator. In this work, the bondedness of $\mu_\Omega^\rho$ is discussed on Herz spaces $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$, where the two main indices are variable exponent. The boundedness of the commutators generated by BOM function, Lipschitz function and parametrized Marcinkiewicz integrals operator is also discussed.

**Keywords:** BMO function, Commutator, Herz space with variable exponent, Lipschitz function, Parametrized Marcinkiewicz integral operator.

## 1. Introduction

Suppose $\mathbb{S}^{n-1}$ for $n \geq 2$ is the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Further suppose that $\Omega$ is a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying $\Omega \in L^1(\mathbb{S}^{n-1})$ and

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0, \text{ where } x' = x/|x|(x \neq 0). \tag{1}$$

For $0 < \rho < n$, the parametrized Marcinkewicz integrals is defined as;

$$\mu_\Omega^\rho(h)(x) = \left(\int_0^\infty |F_{\Omega,t}^\rho(h)(x)|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2},$$

where $F_{\Omega,t}^\rho(h)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h(y)dy,\ t > 0$.

For $m \in \mathbb{N}$, $b \in \text{BMO}(\mathbb{R}^n)$, the higher-order commutator of parametrized Marcinkewicz integral is defined as;

$$[b^m, \mu_\Omega^\rho](h)(x) = \left(\int_0^\infty \left|\int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} [b(x) - b(y)]^m h(y)dy\right|^2 \frac{dt}{t^{2\rho+1}}\right)^{1/2},\ t > 0. \tag{2}$$

It is easy to see that when $\rho = 1$, and $\mu^\rho(h) = \mu^1(h)$, then (2) is the classical Marcinkiewicz integral $\mu(h)$ introduced by Stein in [1]. It has been proved in [1] that if $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$ $(0 < \gamma \leq 1)$ and $\Omega$ is continuous, then the operator $\mu(h)$ is of the type $(q,q)$ for $1 < q \leq 2$ and of the weak type $(1,1)$. Benedek *et al.*, [2] proved that if $\Omega \in C^1(\mathbb{S}^{n-1})$, then $\mu(h)$ it is of type $(q,q)$ for any $1 < q \leq \infty$. The $L^p$ boundedness of the $\mu(h)$ has been studied in [1,3–5].

In 1960, Hörmander [4] introduced the parametrized Marcinkewicz integral operators proved that if $\Omega \in Lip_\gamma(\mathbb{S}^{n-1})$, $0 < \gamma \leq 1$, then it is of strong type $(q,q)$ for $1 < q \leq 2$. Sakamoto and Yabuta [6] proved the boundedness of the operator $\mu^\rho(h)$ on $L^q(\mathbb{R}^n)$. Shi and Jiang [7] considered the weighted $L^q-$boundedness of parametrized Marcinkewicz integral operator and its higher order commutator. Note that the Littlewood-paley $g$-function played very important roles in harmonic analysis and the parameterized Marcinkiewick integral is a special case of the Littlewood-paley $g$-function. Many authors studied properties of $\mu^\rho(h)$ on different function spaces, for examples [8–14].

In the last three decade, the generalized Orlicz-Lebesgue spaces and the corresponding generalized Orlicz-Sobolev spaces have been extensively studied by many researchers. The variable Lebesgue spaces are special cases of generalized orliz spaces which introduced by Nakano in [15] and developed in [16,17]. In addition, for properties of $L^{p(\cdot)}$ spaces we refer to [18–20], and the fundamental paper of Kováčik and Rákosník [21] appeared in 1990. By virtue of this works many function spaces appeared [22–25]. Recently, in 2015, Lijuan and Tao established the Herz spaces with two variable exponents $p(\cdot)$, $q(\cdot)$ in the paper [26].

The main purpose of this work is to discuss the boundedness of parameterized Marcinkiewicz integral and it's higher order commutators with rough kernels on Herz spaces with two variable exponents. The boundedness of higher order commutator generated by BOM function and parameterized Marcinkiewicz integral is also obtained.

Let Y be a measurable set in $\mathbb{R}^n$ with $|Y| > 0$.

**Definition 1.** Let $p(\cdot) : Y \to [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(Y)$ is defined by

$$L^{p(\cdot)}(Y) = \left\{ h \text{ is measurable} : \int_{\Omega} \left( \frac{|h(x)|}{\eta} \right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0 \right\}$$

The space $L^{p(\cdot)}_{loc}(Y)$ is defined by

$$L^{p(\cdot)}_{loc}(Y) = \{\text{h is measurable} : h \in L^{p(\cdot)}(K) \quad \text{for all compact } K \subset Y\}$$

The Lebesgue spaces $L^{p(\cdot)}(Y)$ is a Banach spaces with the norm defined by

$$\|h\|_{L^{p(\cdot)}(Y)} = \inf \left\{ \eta > 0 : \int_{\Omega} \left( \frac{|h(x)|}{\eta} \right)^{p(x)} dx \leq 1 \right\}, \tag{3}$$

We denote

$$p_- = \text{essinf}\{p(x) : x \in Y\}, \qquad p_+ = \text{ess sup}\{p(x) : x \in Y\},$$

then $\mathcal{P}(Y)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let $M$ be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(Y)$ to be the set of all function $p(\cdot) \in \mathcal{P}(Y)$ such that $M$ is bounded on $L^{p(\cdot)}(Y)$.

Now, let us recall the definition of Herz spaces with variable exponents.

**Definition 2.** [26]Let $\alpha \in \mathbb{R}^n$, $q(\cdot)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{h \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \backslash \{0\}) : \|h\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|h\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left\| \{2^{k\alpha}|h\chi_k|\}_{k=0}^{\infty} \right\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \eta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha}|h\chi_k|}{\eta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

**Remark 1.** Let $v \in \mathbb{N}$, $a_v \geq 0$, $1 \leq p_v < \infty$, then

$$\sum_{v=0}^{\infty} a_v \leq \left( \sum_{v=0}^{\infty} a_v \right)^{p_*},$$

where

$$p_* = \begin{cases} \min_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v \leq 1, \\ \max_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v > 1. \end{cases}$$

**Remark 2.** [26]

1. If $q_1(\cdot),\ q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying $(q_1)_+ \leq (q_2)_+$, then $K^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset K^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, $\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.
2. If $q_1(\cdot),\ q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $(q_1)_+ \leq (q_2)_-$, then $\frac{q_2(\cdot)}{q_1(\cdot)} \in \mathcal{P}(\mathbb{R}^n)$ and $\frac{q_2(\cdot)}{q_1(\cdot)} \geq 1$.

By Remark 1, for any $h \in \dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|h\chi_k|}{\eta}\right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|h\chi_k|}{\eta}\right)^{q_1(\cdot)} \right\|^{p_v}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \leq \left\{ \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha}|h\chi_k|}{\eta}\right)^{q_1(\cdot)} \right\|^{p_h}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \right\}^{p_*} \leq 1;$$

where

$$p_v = \begin{cases} \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_-, & \frac{2^{k\alpha}|f\chi_k|}{\eta} \leq 1, \\ \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_+, & \frac{2^{k\alpha}|f\chi_k|}{\eta} > 1, \end{cases}$$

and

$$p_* = \begin{cases} \min_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v \leq 1, \\ \max_{v\in\mathbb{N}} p_v, & \sum_{v=0}^{\infty} a_v > 1. \end{cases}$$

This implies that $\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset \dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Similarly, we get $K^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n) \subset K^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

**Definition 3.** For all $0 < \gamma \leq 1$, the Lipschitz space $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ is defined by

$$\dot{\Lambda}_\gamma(\mathbb{R}^n) = \left\{ h : \|h\|_{\dot{\Lambda}_\gamma(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n;\ x\neq y} \frac{|h(x)-h(y)|}{|x-y|^\gamma} < \infty \right\}.$$

**Definition 4.** The BMO function and BMO norm are defined by

$$\mathrm{BMO}(\mathbb{R}^n) := \left\{ b \in L^1_{loc}(\mathbb{R}^n) : \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} < 0 \right\},$$
$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q:\text{cube}} |Q|^{-1} \int_Q |b(x) - b_Q| dx.$$

From here, we suppose that $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, and $C_k = B_k \backslash B_{k-1}, \chi_k = \chi_{C_k},\ \ k \in \mathbb{Z}$.

## 2. Preliminary Lemmas

**Proposition 1.** *[27] Let a function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$. If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies*

$$|p(x) - p(y)| \leq \frac{-C}{Log(|x-y|)}; \qquad |x-y| \leq 1/2, \tag{4}$$

*and*

$$|p(x) - p(y)| \leq \frac{C}{Log(e+|x|)}; \qquad |y| \geq |x|, \tag{5}$$

*then $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.*

**Lemma 1.** *[21] (Generalized Hölder Inequality) Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then*

1. *for every $h \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^n} |h(x)g(x)|dx \le C\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$, where $C_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$;*
2. *for every $h \in L^{p_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, when $\frac{1}{p(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{p_1(\cdot)}$, we have $\|h(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|g(x)\|_{L^{p_2}(\mathbb{R}^n)}\|h(x)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}$, where $C_{p_1,p_2} = [1 + \frac{1}{p_{1-}} - \frac{1}{p_{1+}}]^{\frac{1}{p_-}}$.*

**Lemma 2.** *[18,19] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. If there exists a positive constants $C$, $\delta_1$, $\delta_2$ such that $\delta_1,\ \delta_2 < 1$, then, for all balls $B \subset \mathbb{R}^n$ and all measurable subset $R \subset B$, we have*

$$\frac{\|\chi_R\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|R|}{|B|},\quad \frac{\|\chi_R\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_u(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{\delta_2},\quad \frac{\|\chi_R\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|R|}{|B|}\right)^{\delta_1}.$$

**Lemma 3.** *[28] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for any balls $B$ in $\mathbb{R}^n$, we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

**Lemma 4.** *[29] Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $i, j \in \mathbb{Z}$ with $i < j$, then we have*

1. *$C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$;*
2. *$\|(b - b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(j - i)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.*

**Lemma 5.** *[26] Let $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $h \in L^{p(\cdot)q(\cdot)}$, then*

$$\min(\|h\|^{q_+}_{L^{p(\cdot)q(\cdot)}},\ \|h\|^{q_-}_{L^{p(\cdot)q(\cdot)}}) \le \||h|^{q(\cdot)}\|_{L^{p(\cdot)}} \le \max(\|h\|^{q_+}_{L^{p(\cdot)q(\cdot)}},\ \|h\|^{q_-}_{L^{p(\cdot)q(\cdot)}}).$$

**Lemma 6.** *[30] Let $a > 0,\ 0 < d \le s,\ 1 \le s \le \infty$ and $\frac{-sn+(n-1)d}{s} < v < \infty$, then*

$$\left(\int_{|y|\le a|x|} |y|^v|\Omega(x-y)|^d\, dy\right)^{1/d} \le C|x|^{(v+n)/d}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}.$$

**Lemma 7.** *[31] Let the variable exponent $\tilde{q}(\cdot)$ is defined by $\frac{1}{p(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{q}$ $(x \in \mathbb{R}^n)$, then we have*

$$\|hg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|g\|_{L^q(\mathbb{R}^n)}\|h\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.$$

**Lemma 8.** *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\Omega \in L^s(\mathbb{S}^{n-1})$ and $0 < \rho < n$. If there exists a constant $C > 0$ independent of $h$, then $\mu^\rho_\Omega$ is bounded from $L^{p(\cdot)}$ to it self.*

**Lemma 9.** *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $m \in \mathbb{N}$. Further let that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\Omega \in L^s(\mathbb{S}^{n-1})$ and $0 < \rho < n$. If there exists a constant $C > 0$ independent of $h$, then $[b^m, \mu^\rho_\Omega]$ is bounded from $L^{p(\cdot)}$ to itself.*

**Lemma 10.** *Let $b \in \dot{\Lambda}_\gamma(\mathbb{R}^n)$, $0 < \gamma \le 1$, $m \in \mathbb{N}$ and $0 < \rho < n$. If $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies (4) and (5) in Proposition 1 with $q_1^+ < n/\gamma$, $1/q_1(x) - 1/q_2(x) = \gamma/n$, $\Omega \in L^s(\mathbb{S}^{n-1})(s > q_2^+)$ with $1 \le r' < q_2^-$. Then the commutator $[b^m, \mu^\rho_\Omega]$ is bounded from $L^{q_1(\cdot)}(\mathbb{R}^n)$ to $L^{q_2(\cdot)}(\mathbb{R}^n)$.*

**Lemma 11.** *[32] Let $p(\cdot) \in \mathcal{P}(\Omega)$ abd $h : \Omega \times \Omega \to \mathbb{R}$ is a measurable function (with respect to product measure) such that, $y \in \Omega$, $h(\cdot, y) \in L^{p(\cdot)}(\Omega)$, then we have*

$$\left\|\int_\Omega h(\cdot,y)dy\right\|_{L^{p(\cdot)}(\Omega)} \le C\int_\Omega \|h(\cdot,y)\|_{L^{p(\cdot)}(\Omega)}\, dy.$$

## 3. Main Results

**Theorem 1.** *Let $0<\rho<n$, $0<v\leq 1$. Suppose that $p_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\Omega\in L^s(\mathbb{S}^{n-1})$, $s>(p_1')_+$ and $q_1(\cdot)$, $q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $(q_2)_-\geq(q_1)_+$. If $-n\delta_1-v-n/s<\alpha<n\delta_2-v-n/s$ with $\delta_1$, $\delta_2$ as defined in Lemma 2, then the operator $\mu_\Omega^\rho$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$ and from $\left(K_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)\right)$ to $\left(K_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)\right)$.*

**Proof.** Let $h(x)\in\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$. Rewrite $h(x)=\sum_{j=-\infty}^{\infty}h(x)\chi_j=\sum_{j=-\infty}^{\infty}h_j(x)$. From Definition 2, we have

$$\|\mu_\Omega^\rho(h)\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\mu_\Omega^\rho(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1\right\}.$$

Since

$$\left\|\left(\frac{2^{k\alpha}|\mu_\Omega^\rho(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{\infty}\mu_\Omega^\rho(h_j)\chi_k|}{\sum_{i=1}^{3}\eta_{1i}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$
$$\leq\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}\mu_\Omega^\rho(h_j)\chi_k|}{\eta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}+\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}\mu_\Omega^\rho(h_j)\chi_k|}{\eta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}+\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}\mu_\Omega^\rho(h_j)\chi_k|}{\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},$$

where

$$\begin{aligned}\eta_{11}&=\left\|\left\{2^{k\alpha}|\sum_{j=-\infty}^{k-2}\mu_\Omega^\rho(h_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},\\ \eta_{12}&=\left\|\left\{2^{k\alpha}|\sum_{j=k-2}^{k+2}\mu_\Omega^\rho(h_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},\\ \eta_{13}&=\left\|\left\{2^{k\alpha}|\sum_{j=k+2}^{\infty}\mu_\Omega^\rho(h_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},\end{aligned}$$

and

$$\eta=\eta_{11}+\eta_{12}+\eta_{13}=\sum_{i=1}^{3}\eta_{1i}.$$

Thus,

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\mu_\Omega^\rho(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq C.$$

Meanwhile,

$$\|\mu_\Omega^\rho(h)\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}\leq C\eta=C\sum_{i=1}^{3}\eta_{1i}.$$

To show Theorem 1, we only need to estimate $\eta_{11},\eta_{12}$ and $\eta_{13}\leq C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$. To do this, denote $\eta_{10}=\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$.

**Step 1.** For $\eta_{12}$. From Lemma 5, we get

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}\mu_\Omega^\rho(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}&\leq\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}\mu_\Omega^\rho(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}^{(q_2^1)_k}\\ &\leq\sum_{k=-\infty}^{\infty}\left(\left\|\frac{2^{k\alpha}|\sum_{j=k-2}^{k+2}\mu_\Omega^\rho(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\right)^{(q_2^1)_k},\end{aligned}\tag{6}$$

where

$$(q_2^1)_k = \begin{cases} (q_2)_-, & \left\| \left( \frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} \mu_\Omega^\rho(h_j)\chi_k|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le 1, \\ (q_2)_+, & \left\| \left( \frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} \mu_\Omega^\rho(h_j)\chi_k|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1. \end{cases}$$

So, by using the Lemma 6, Remark 2 and $h(x) \in \dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$, we have $\left\| \frac{2^{k\alpha}|h\chi_k|}{\eta_{10}} \right\|_{L^{p_1(\cdot)}} \le 1$ and $\sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha}|h\chi_k|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \le 1$. Hence

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha} |\sum_{j=k-2}^{k+2} \mu_\Omega^\rho(h_j)(h_j)\chi_k|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} &\le C \sum_{k=-\infty}^{\infty} \left( \sum_{j=k-2}^{k+2} \left\| \frac{2^{k\alpha}|h_j|}{\eta_{10}} \right\|_{L^{p_1(\cdot)}} \right)^{(q_2^1)_k} \\ \le C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha}|h\chi_k|}{\eta_{10}} \right\|_{L^{p_1(\cdot)}}^{(q_2^1)_k} \le C \sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha}|h\chi_k|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1)_+}} &\le C \left\{ \sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha}|h\chi_k|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right\}^{q_*} \le C. \end{aligned} \tag{7}$$

Which, together with $(p_1)_+ \le (p_2)_- \le (q_2^1)_k$ and $q_* = \min_{k \in N} \frac{(q_2^1)_k}{(q_1)_+}$ gives;

$$\eta_{12} \le C\eta_{10} \le C\|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}. \tag{8}$$

**Step 2.** Now, let us deal with $\eta_{11}$. Since

$$\begin{aligned} |\mu_\Omega^\rho(h_j)(x)| &:= \left( \int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h_j(y) dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} \\ &\le \left( \int_0^{|x|} \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h_j(y) dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} \\ &\quad + \left( \int_{|x|}^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h_j(y) dy \right|^2 \frac{dt}{t^{2\rho+1}} \right)^{1/2} \\ &:= \eta'_{11} + \eta''_{11}. \end{aligned}$$

Now we estimate $\eta'_{11}$ and $\eta''_{11}$. For $\eta'_{11}$, note that $x \in A_k$, $y \in A_j$ and $j \le k-2$. Since $|x-y| \sim |x|$ so by virtue of the Mean Value Theorem, we have

$$\left| \frac{1}{|x-y|^{2\rho}} - \frac{1}{|x|^{2\rho}} \right| \le C \frac{|y|}{|x-y|^{2\rho+1}}. \tag{9}$$

Substituting the inequality (9) into $\eta'_{11}$ and by virtue of Minkowski's inequality, we deduced that

$$\begin{aligned} \eta'_{11} &\le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |h_j(y)| \left( \int_{|x-y|}^{|x|} \frac{dt}{t^{2\rho+1}} \right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |h_j(y)| \left| \frac{1}{|x-y|^{2\rho}} - \frac{1}{|x|^{2\rho}} \right|^{1/2} dy \\ &\le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |h_j(y)| \frac{|y|^{1/2}}{|x-y|^{\rho+1/2}} dy \le C \frac{2^{j/2}}{|x|^{n+1/2}} \int_{A_j} |\Omega(x-y)| \, |h_j(y)| dy \\ &\le C 2^{j/2} 2^{-k(n+1/2)} \int_{A_j} |\Omega(x-y)| \, |h_j(y)| dy \le C 2^{(j-k)/2} 2^{-nk} \int_{A_j} |\Omega(x-y)| \, |h_j(y)| dy. \end{aligned} \tag{10}$$

Similarly, we obtain

$$\begin{aligned}\eta''_{11} &\leq C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|h_j(y)|\left(\int_{|x|}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{1/2}dy \leq C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|h_j(y)|\left(\frac{1}{|x|^{2\rho}}\right)^{1/2}dy\\ &\leq C\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n}}|h_j(y)|dy \leq C2^{-nk}\int_{A_j}|\Omega(x-y)|\,|h_j(y)|dy. \end{aligned} \tag{11}$$

Combining the inequality (11) with Lemma 1, we get

$$|\mu^{\rho}_{\Omega}(h_j)(x)| \leq C2^{-nk}\int_{A_j}|\Omega(x-y)|\,|h_j(y)|dy \leq C2^{-nk}\|\left(\Omega(x-\cdot)\right).\chi_{B_j}\|_{L^{p'_1(\cdot)}}\|h_j\|_{L^{p_1(\cdot)}}. \tag{12}$$

Now, consider $\tilde{p}'_1(\cdot) > 1$ and $1/p'_1(x) = 1/\tilde{p}'_1(x) + 1/s$. Since $s > (p'_1)_+$, so by virtue of Lemma 1 and Lemma 8, we get

$$\begin{aligned}&\|\left(\Omega(x-\cdot)\right).\chi_{B_j}\|_{L^{p'_1(\cdot)}} \leq \|\Omega(x-\cdot)\|_{L^s}\|\chi_{B_j}\|_{L^{\tilde{p}'(\cdot)}} \leq 2^{-jv}\left(\int_{A_j}|y|^{sv}|\Omega(x-y)|^s dy\right)^{1/s}\|\chi_{B_j}\|_{L^{\tilde{p}'_1(\cdot)}}\\ &\leq 2^{-jv}2^{k(v+n/s)}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{\tilde{p}'_1(\cdot)}} \leq 2^{-jv}2^{k(v+n/s)}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}/|B_j|^{1/s}\\ &\leq 2^{(k-j)(v+n/s)}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}. \end{aligned} \tag{13}$$

By using (12), (13), Lemmas 1, 2, 3, 5 and $\left\|\frac{2^{j\alpha}|h\chi_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)q_1}} \leq 1$, we get

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}\mu^{\rho}_{\Omega}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C\sum_{k=-\infty}^{\infty}\left(\left\|\frac{2^{k\alpha}|\sum_{j=}^{\infty}\mu^{\rho}_{\Omega}(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\ &\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}2^{(k-j)(v+n/s)}\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^2)_k}\\ &\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{(k-j)(v+n/s)}\left\|\frac{h\chi_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^2)_k}\\ &\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{(k-j)(v+n/s)}2^{-j\alpha}\left\|\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^2)_k}\\ &\leq C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha+v+n/s-n\delta_2)}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1)_+}}_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right\}^{(q_2^2)_k}, \end{aligned} \tag{14}$$

where

$$(q_2^2)_k = \begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}\mu^{\rho}_{\Omega}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}\mu^{\rho}_{\Omega}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} > 1.\end{cases}$$

Which, together with $(q_1)_+ < 1$ and $(p_1)_+ \leq (p_2)_- \leq (q_2^2)_k$ gives;

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}\mu^{\rho}_{\Omega}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} &\leq C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha+v+n/s-n\delta_2)}\right\}^{q_*}\\ &\leq C, \end{aligned} \tag{15}$$

where $q_* = \min\limits_{k\in\mathbb{N}}\frac{(q_2^1)_k}{(q_1)_+}$.

Since $\alpha < n\delta_2 - (v + n/s)$, so if $(q_1)_+ \geq 1$ and $(q_2^2)_k \geq (q_2)_- \geq (q_1)_+ \geq 1$ then by using Remark 2and applying the generalized Hölder's inequality, we get

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha} |\sum_{j=-\infty}^{k-2} \mu_{\Omega}^{\rho}(h_j)\chi_k|}{\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \\
&\leq C \sum_{k=-\infty}^{\infty} \left\{ \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha+v+n/s-n\delta_2)(q_1)_+/2} \left\| \left( \frac{|2^{j\alpha} h\chi_j|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)q_1(\cdot)}} \right\}^{\frac{(q_2^2)_k}{(q_1)+}} \\
&\quad \times \left( \sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha+v+n/s-n\delta_2)((q_1)_+)'/2} \right)^{\frac{(q_2^2)_k}{((q_1)+)'}} \\
&\leq C \left\{ \sum_{j=-\infty}^{\infty} \left\| \left( \frac{|2^{j\alpha} h\chi_j|}{\eta_{10}} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)q_1(\cdot)}} \sum_{k=j+2}^{\infty} 2^{(k-j)(\alpha+v+n/s-n\delta_2)(q_1)_+/2} \right\}^{q_*} \\
&\leq C, \qquad (16)
\end{aligned}$$

where $q_* = \min_{k \in \mathbb{N}} \frac{(q_2^2)_k}{(q_1)_+}$. Hence we have

$$\eta_{11} \leq C\eta_{10} \leq C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}. \qquad (17)$$

**Step 3.** Finally, we estimate $\eta_{13}$. For each $x \in A_j$ and $j \geq k + 2$, we have

$$\begin{aligned}
|\mu_{\Omega}^{\rho}(h_j)(x)| &:= \left( \int_0^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h_j(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2} \\
&\leq \left( \int_0^{|y|} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h_j(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2} \\
&\quad + \left( \int_{|y|}^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} h_j(y)\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2} \\
&:= \eta_{13}' + \eta_{13}''.
\end{aligned}$$

The estimates of $\eta_{13}'$ and $\eta_{13}''$ can be obtained similarly as that of $\eta_{11}'$ and $\eta_{11}''$ in Step 2 and we get

$$\eta_{13}' \leq C2^{(j-k)/2}2^{-jn} \int_{A_j} |\Omega(x-y)|\,|h_j(y)|\mathrm{d}y, \qquad (18)$$

and

$$\eta_{13}'' \leq C2^{-jn} \int_{A_j} |\Omega(x-y)|\,|h_j(y)|\mathrm{d}y. \qquad (19)$$

Thus, we have

$$|\mu_{\Omega}^{\rho}(h_j)(x)| \leq C2^{-jn} \int_{A_j} |\Omega(x-y)|\,|h_j(y)|\mathrm{d}y \leq C2^{-jn}\|\left(\Omega(x-\cdot)\right).\chi_{B_j}\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}}. \qquad (20)$$

Substituting (13) into (20), together with Lemmas 1, 2, 3, 5 and $\left\| \frac{2^{j\alpha}|h\chi_j|}{\eta_{10}} \right\|_{L^{p_1(\cdot)q_1(\cdot)}} \leq 1$, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}\mu_{\Omega}^{\rho}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}\mu_{\Omega}^{\rho}(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}^{(q_2^3)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}2^{(k-j)(v+n/s)}\times\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\right)^{(q_2^3)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}2^{(k-j)(v+n/s)}\left\|\frac{h\chi_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}|B_j|\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^3)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(k-j)(v+n/s)}\left\|\frac{h\chi_j}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^3)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(k-j)(v+n/s)}2^{-j\alpha}\left\|\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{j_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\right)^{(q_2^3)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha+v+n/s+n\delta_{12})}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right\}^{(q_2^3)_k}, \tag{21}$$

where

$$(q_2^3)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}\mu_{\Omega}^{\rho}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^{\infty}\mu_{\Omega}^{\rho}(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}> 1.\end{cases}$$

From above and by an argument similar to that of Step 2, we conclude

$$\eta_{13}\leq C\eta_{10}\leq C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}. \tag{22}$$

The proof is completed. □

**Theorem 2.** *Suppose $b\in \mathrm{BMO}(\mathbb{R}^n)$, $m\in\mathbb{N}$, $0<\rho<n$, $0<v\leq 1$. Further suppose that $p_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\Omega\in L^s(\mathbb{S}^{n-1})$, $s>(p_1')_+$ and $q_1(\cdot)$, $q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $(q_2)_-\geq(q_1)_+$. If $-n\delta_1-v-n/s<\alpha<n\delta_2-v-n/s$ with $\delta_1$, $\delta_2$ as defined in Lemma 2. Then the operator $[b^m,\mu_{\Omega}^{\rho}]$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$ and $\left(K_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)\right)$ to $\left(K_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)\right)$.*

**Proof.** Let $h(x)\in\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$, $b\in\mathrm{BMO}(\mathbb{R}^n)$. We may write $h(x)=\sum_{j=-\infty}^{\infty}h(x)\chi_j=\sum_{j=-\infty}^{\infty}h_j(x)$. By definition of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$, we have

$$\|[b^m,\mu_{\Omega}^{\rho}](h)\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|[b^m,\mu_{\Omega}^{\rho}](h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1\right\}.$$

Since

$$\left\|\left(\frac{2^{k\alpha}|[b^m,\mu_{\Omega}^{\rho}](h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_{\Omega}^{\rho}](h_j)\chi_k|}{\sum_{i=1}^{3}\eta_{2i}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq \left\| \left( \frac{2^{k\alpha} | \sum_{j=-\infty}^{\infty} [b^m, \mu_\Omega^\rho](h_j)\chi_k|}{\eta_{21}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} + \left\| \left( \frac{2^{k\alpha} | \sum_{j=k-2}^{k+2} [b^m, \mu_\Omega^\rho](h_j)\chi_k|}{\eta_{22}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$
$$+ \left\| \left( \frac{2^{k\alpha} | \sum_{j=k+2}^{\infty} [b^m, \mu_\Omega^\rho](h_j)\chi_k|}{\eta_{23}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}.$$

Let

$$\eta_{21} = \left\| \left\{ 2^{k\alpha} | \sum_{j=-\infty}^{k-2} [b^m, \mu_\Omega^\rho](h_j)\chi_k| \right\}_{k=-\infty}^{\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},$$
$$\eta_{22} = \left\| \left\{ 2^{k\alpha} | \sum_{j=k-2}^{k+2} [b^m, \mu_\Omega^\rho](h_j)\chi_k| \right\}_{k=-\infty}^{\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},$$
$$\eta_{23} = \left\| \left\{ 2^{k\alpha} | \sum_{j=k+2}^{\infty} [b^m, \mu_\Omega^\rho](h_j)\chi_k| \right\}_{k=-\infty}^{\infty} \right\|_{\ell^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

where we put

$$\eta = \eta_{21} + \eta_{22} + \eta_{23} = \sum_{i=1}^{3} \eta_{2i}.$$

Hence,

$$\sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha} |[b^m, \mu_\Omega^\rho](h)\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C.$$

So, it follows that

$$\|[b^m, \mu_\Omega^\rho](h)\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} \leq C\eta = C\sum_{i=1}^{3} \eta_{1i}.$$

Hence, $\eta_{21}, \eta_{22}$ and $\eta_{23} \leq C\|b\|_*\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$. Denoting that $\eta_{10} = C\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$.

**S**tep 1. We estimate $\eta_{22}$. The proof of Theorem 2 is the same to that of Theorem 1 and we use the similar notation as in the proof $\eta_{12}$ of Theorem 1. By Lemma 5 and $\left(L^{p(\cdot)}(\mathbb{R}^n), L^{p(\cdot)}(\mathbb{R}^n)\right)$-boundedness of the operators $[b^m, \mu_\Omega^\rho]$ , we directly arrive at

$$\sum_{k=-\infty}^{\infty} \left\| \left( \frac{2^{k\alpha} | \sum_{j=k-2}^{k+2} [b^m, \mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}\|b\|_*} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C,$$

which, implies that

$$\eta_{21} \leq C\eta_{10}\|b\|_* \leq C\|b\|_*\|h\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}. \tag{23}$$

**S**tep 2. Next we estimate $\eta_{21}$. Since

$$\begin{aligned} |[b^m, \mu_\Omega^\rho](h_j)(x)| &:= \left( \int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} [b(x)-b(y)]^m h_j(y) \mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2} \\ &\leq \left( \int_0^{|x|} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} [b(x)-b(y)]^m h_j(y) \mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2} \\ &+ \left( \int_{|x|}^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} [b(x)-b(y)]^m h_j(y) \mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2} \\ &:= '_{22} + ''_{22}. \end{aligned}$$

Observe that $|x-y| \approx |x|$ for each $x \in A_k, y \in A_j$ and $j \leq k-2$. From (9) and applying the Minkowski's and the generalized Hölder's inequality, we get

$$\begin{aligned}
{}'_{22} &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}[b(x)-b(y)]^m|h_j(y)| \left(\int_{|x-y|}^{|x|} \frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{1/2} \mathrm{d}y \\
&\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}[b(x)-b(y)]^m|h_j(y)| \left|\frac{1}{|x-y|^{2\rho}} - \frac{1}{|x|^{2\rho}}\right|^{1/2} \mathrm{d}y \\
&\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}[b(x)-b(y)]^m|h_j(y)| \frac{|y|^{1/2}}{|x-y|^{\rho+1/2}}\mathrm{d}y \\
&\leq C\frac{2^{j/2}}{|x|^{n+1/2}}\left\{[b(x)-b_{B_j}]^m\int_{A_j}|\Omega(x-y)|\,|h_j(y)|\mathrm{d}y + \int_{A_j}|\Omega(x-y)|\,[b_{B_j}-b(y)]^m|h_j(y)|\mathrm{d}y\right\} \\
&\leq C2^{j/2}2^{-k(n+1/2)}\left\{[b(x)-b_{B_j}]^m\|\left(\Omega(x-\cdot)\right).\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|h_j\|_{L^{p_1(\cdot)}}\right. \\
&\quad \left.+\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m(h_j).\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|h_j\|_{L^{p_1(\cdot)}}\right\}. \qquad (24)
\end{aligned}$$

Similarly, we consider ${}''_{22}$

$$\begin{aligned}
{}''_{22} &\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}[b(x)-b_{B_j}]^m|h_j(y)| \left(\int_{|x|}^{\infty} \frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{1/2} \mathrm{d}y \\
&\leq C\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}[b(x)-b_{B_j}]^m|h_j(y)| \left(\frac{1}{|x|^{2\rho}}\right)^{1/2} \mathrm{d}y \\
&\leq C2^{-nk}\left\{[b(x)-b_{B_j}]^m\int_{A_j}|\Omega(x-y)|\,|h_j(y)|\mathrm{d}y + \int_{A_j}|\Omega(x-y)|\,[b_{B_j}-b(y)]^m|h_j(y)|\mathrm{d}y\right\} \\
&\leq C2^{-nk}\left\{[b(x)-b_{B_j}]^m\|\left(\Omega(x-\cdot)\right)\cdot\chi_j(\cdot)\|_{L^{p_1'(\cdot)}}\|h_j\|_{L^{p_1(\cdot)}}\right. \\
&\quad \left.+\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m(h_j)\cdot\chi_j(\cdot)\|_{L^{p_1'(\cdot)}}\|h_j\|_{L^{p_1(\cdot)}}\right\}. \qquad (25)
\end{aligned}$$

Therefore,

$$\begin{aligned}
|[b^m,\mu_\Omega^\rho](h_j)(x)| &\leq C2^{-nk}\left\{[b(x)-b_{B_j}]^m\|\left(\Omega(x-\cdot)\right)\cdot\chi_j(\cdot)\|_{L^{p_1'(\cdot)}}\|h_j\|_{L^{p(\cdot)}}\right. \\
&\quad \left.+\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m\cdot\chi_j(\cdot)\|_{L^{p_1'(\cdot)}}\|h_j\|_{L^{p_1(\cdot)}}\right\}. \qquad (26)
\end{aligned}$$

By (13) and Lemmas 6 and 7, we get

$$\begin{aligned}
&\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m\cdot\chi_j(\cdot)\|_{L^{p_1'(\cdot)}} \leq \|\Omega(x-\cdot)\cdot\chi_j(\cdot)\|_{L^s}\|(b_{B_j}-b(\cdot))^m\cdot\chi_j(\cdot)\|_{L^{\tilde{p}_1'(\cdot)}} \\
&\leq 2^{-jv}2^{k(v+n/s)}\|b\|_*^m\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{\tilde{p}_1'(\cdot)}} \leq 2^{(k-j)(v+n/s)}\|b\|_*^m\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}. \qquad (27)
\end{aligned}$$

From this, we deduced

$$\begin{aligned}
\|[b^m,\mu_\Omega^\rho](h_j)(x)\cdot\chi_{B_k}\|_{L^{p_1(\cdot)}} &\leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}2^{-nk}2^{(k-j)(v+n/s)}\|h_j\|_{L^{p_1(\cdot)}}\|(b(\cdot)-b_{B_j})^m\cdot\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \\
&\quad +C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}2^{-nk}2^{(k-j)(v+n/s)}\|b\|_*^m\|h_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}. \qquad (28)
\end{aligned}$$

Applying Lemmas 1, 3, 4 and 5, we have

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}\|b\|_*^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}\|b\|_*^m}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{(q_2^2)_k} \\
&\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}2^{(k-j)(v+n/s)}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\frac{1}{\|b\|_*^m}\|(b(\cdot)-b_{B_j})^m\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\right)^{(q_2^2)_k}
\end{aligned}$$

$$+C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}(k-j)^m 2^{-kn}2^{(k-j)(v+n/s)}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\right)^{(q_2^2)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}(k-j)^m 2^{-kn}2^{(k-j)(v+n/s)}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\right)^{(q_2^2)_k}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=-\infty}^{k-2}(k-j)^m 2^{(k-j)(v+n/s)}2^{-j\alpha}\left\|\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\right)^{(q_2^2)_k}.$$

Now, by Lemma 2, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{\infty}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-2}(k-j)^m 2^{(k-j)(\alpha+v+n/s-n\delta_2)}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right\}^{(q_2^2)_k}, \tag{29}$$

where

$$(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}> 1.\end{cases}$$

So, together with $(q_1)_+<1$, $(p_1)_+\leq(p_2)_-\leq(q_2^2)_k$, along with Remark 1, gives

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$
$$\leq C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}(k-j)^m 2^{(k-j)(\alpha+v+n/s-n\delta_2)}\right\}^{q_*}$$
$$\leq C, \tag{30}$$

where $q_*=\min\limits_{k\in N}\frac{(q_2^2)_k}{(q_1)_+}$.

If $(q_1)_+\leq 1$, then by Hölder's inequality and Remark 1, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$
$$\leq C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha+v+n/s-n\delta_2)(q_1)_+/2}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\right\}^{\frac{(q_2^2)_k}{(q_1)_+}}$$
$$\times\left(\sum_{j=-\infty}^{k-2}(k-j)^m 2^{(k-j)(\alpha+v+n/s-n\delta_2)((q_1)_+)'/2}\right)^{\frac{(q_2^2)_k}{((q_1)_+)'}}$$
$$\leq C\left\{\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha+v+n/s-n\delta_2)(q_1)_+/2}\right\}^{q_*}\leq C, \tag{31}$$

where $q_* = \min_{k\in N} \frac{(q_2^2)_k}{(q_1)_+}$. This implies that

$$\eta_{21} \le C\eta_{10}\|b\|_* \le C\|b\|_*\|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}. \tag{32}$$

Finally we estimate $\eta_{23}$. For any $x \in A_j$, $j \ge k+2$, by the same argument as in $\eta_{21}$, we obtain

$$\begin{aligned} |[b^m, \mu^\rho_\Omega](h_j)(x)| \quad &:= \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m h_j(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{1/2} \\ &\le \left(\int_0^{|y|} \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m h_j(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{1/2} \\ &+ \left(\int_{|y|}^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}}[b(x)-b(y)]^m h_j(y)\mathrm{d}y\right|^2 \frac{\mathrm{d}t}{t^{2\rho+1}}\right)^{1/2} \\ &:= '_{23} + ''_{23}. \end{aligned}$$

Noticing that $j \ge k+2$. To estimate $\eta'_{23}$ and $\eta''_{23}$ we will use same method as that of $\eta'_{21}$ and $\eta''_{21}$ in Step 2. Since

$$\begin{aligned} '_{23} \quad \le \quad & C2^{(k-j)/2}2^{-jn}\Big\{[b(x)-b_{B_j}]^m\|\,(\Omega(x-\cdot))\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}} \\ & +\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m(h_j)\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}}\Big\} \end{aligned} \tag{33}$$

and

$$\begin{aligned} ''_{23} \quad \le \quad & C2^{-jn}\Big\{[b(x)-b_{B_j}]^m\|\,(\Omega(x-\cdot))\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}} \\ & +\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m(h_j)\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}}\Big\}. \end{aligned} \tag{34}$$

Thus,

$$\begin{aligned} |[b^m,\mu^\rho_\Omega](h_j)(x)| \quad \le \quad & C2^{-jn}\Big\{[b(x)-b_{B_j}]^m\|\,(\Omega(x-\cdot))\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}} \\ & +\|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}}\|h_j\|_{L^{p(\cdot)}}\Big\}. \end{aligned} \tag{35}$$

From (13), by using Lemma 7 and Lemma 2, we get

$$\begin{aligned} \|\Omega(x-\cdot)(b_{B_j}-b(\cdot))^m\cdot\chi_j(\cdot)\|_{L^{p'(\cdot)}} \quad &\le \quad \|\Omega(x-\cdot)\|_{L^s}\|(b_{B_j}-b(\cdot))^m\cdot\chi_j(\cdot)\|_{L^{\tilde{p}'(\cdot)}} \\ &\le \quad 2^{-jv}2^{k(v+n/s)}\|b\|^m_*\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|\chi_{B_j}\|_{L^{\tilde{p}'(\cdot)}}. \end{aligned} \tag{36}$$

Hence, we plug the inequality (36) into (35) and obtain

$$\begin{aligned} \|[b^m,\mu^\rho_\Omega](h_j)(x)\chi_{B_k}\|_{L^{p_1(\cdot)}} \quad \le \quad & C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}2^{-jn}2^{(k-j)(v+n/s)}\|h_j\|_{L^{p_1(\cdot)}}\|(b(\cdot)-b_{B_j})^m\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}} \\ & +C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}2^{-jn}2^{(k-j)(v+n/s)}\|b\|^m_*\|h_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}. \end{aligned} \tag{37}$$

By Lemma 5 and the above inequality, we have

$$\begin{aligned} &\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|\sum_{j=k+2}^\infty[b^m,\mu^\rho_\Omega](h_j)\chi_k|}{\eta_{10}\|b\|^m_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \le C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}|\sum_{j=k+2}^\infty[b^m,\mu^\rho_\Omega](h_j)\chi_k|}{\eta_{10}\|b\|^m_*}\right\|^{(q_2^2)_k}_{L^{p_1(\cdot)}} \\ &\le C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-jn}2^{(k-j)(v+n/s)}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\frac{1}{\|b\|^m_*}\|(b(\cdot)-b_{B_j})^m\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}\right)^{(q_2^2)_k} \end{aligned}$$

$$+C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}(j-k)^m2^{-jn}2^{(k-j)(v+n/s)}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\right)^{(q_2^2)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}(j-k)^m2^{-jn}2^{(k-j)(v+n/s)}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\right)^{(q_2^2)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(2^{k\alpha}\sum_{j=k+2}^{\infty}(j-k)^m2^{(k-j)(v+n/s)}2^{-j\alpha}\left\|\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right\|_{L^{p_1(\cdot)}}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\right)^{(q_2^2)_k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left\{\sum_{j=k+2}^{\infty}(j-k)^m2^{(k-j)(\alpha+v+n/s+n\delta_{12})}\left\|\left(\frac{|2^{j\alpha}h\chi_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}^{\frac{1}{(q_1)+}}\right\}^{(q_2^3)_k} \tag{38}$$

where

$$(q_2^3)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha}|\sum_{j=-\infty}^{k-2}[b^m,\mu_\Omega^\rho](h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}> 1.\end{cases}$$

Hence, by the similar argument to Theorem 1, we arrive at $\eta_{23}\leq C\eta_{10}\|b\|_*\leq C\|b\|_*\|h\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$. This completes the proof. □

**Theorem 3.** *Let $b\in\dot{\Lambda}_\gamma(\mathbb{R}^n)$, $0<\gamma\leq 1$, $m\in\mathbb{N}$, $0<\rho<n$, $0<v\leq 1$. Suppose that $q_1^+<n/m\gamma$, $1/q_1(x)-1/q_2(x)=m\gamma/n$, $\Omega\in L^s(\mathbb{S}^{n-1})(s>q_2^+)$ with $1\leq r'<q_2^-$, $p_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\Omega\in L^s(\mathbb{S}^{n-1})$, $s>(p_1')_+$ and $q_1(\cdot)$, $q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $(q_2)_-\geq(q_1)_+$. If $-n\delta_1-v-n/s<\alpha<n\delta_2-v-n/s$ with $\delta_1$, $\delta_2$ as defined in Lemma 2, then the operator $[b^m,\mu_\Omega^\rho]$ is bounded from $\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n$ and from $\left(K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$ to $\left(K^{\alpha,q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)\right)$.*

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