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Laguerre collocation method for solving higher order linear boundary value problems

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Abstract: Collocation methods are efficient approximate methods developed by utilizing suitable set of functions known as trial or basis functions. These methods are used for solving differential equations, integral equations and integro-differential equations, etc. In this study, the Laguerre polynomial of degree 10 is used as a basis function to propose a collocation method for solving higher order linear ordinary differential equations. Four examples on $4th$, $6th$, $8th$ and $10th$ order ordinary differential equations are selected to illustrate the effectiveness of the method. The numerical results show that the proposed collocation method is easy and straightforward to implement, nevertheless, it is very accurate.

Keywords: Laguerre polynomials, collocation method, orthogonal polynomials, boundary value problems.

1. Introduction

Higher order boundary value problems (BVPs) in ordinary differential equations (ODEs) are important tools for modelling different physical phenomena in sciences and engineering [1–3]. Although many ordinary differential equations especially linear ODEs have known analytical solutions, searching for numerical solutions is important because they provide reliable approximations to problems that are difficult to solve analytically [4]. In many situations, sound mathematical theories are often required for analysis, however, the closed form solutions may be too complicated, thus approximate solutions may be preferred [5].

Over the years, researchers have developed many numerical methods besides collocation methods for handling higher order BVPs. The least squares solutions of $8th$ order boundary value problems using the theory of functional connections was developed by [3]. Similarly, [6] considered the numerical solution of $8th$ order boundary value problems which arise in magnetic fields and cylindrical shells. In the same vein, [7] used the Legendre Galerkin method for the numerical solution of 8th order linear boundary value problems. [8] approximated the solution of some mth order linear boundary value problems where $2 \leq m \leq 9$ by the use of a numerical method constructed with "Tchebychev" polynomial. The approximation of linear 10th order boundary value problems via polynomial and non-polynomial cubic spline techniques was considered by [9]. [10] applied the optimal homotopy asymptotic method to $8th$ order initial and boundary value problems. [11] developed a continuous k-step linear multistep method (LMM) that was utilized to generate finite difference methods which were assembled and applied as simultaneous numerical integrators to solve $4th$ order initial value and boundary value problems directly. [12] proposed a method for the numerical solution of special $4th$ order BVPs via modified decomposition method.

Collocation methods using splines, polynomials and orthogonal polynomials have been developed and applied for the solution of higher order BVPs. [1] proposed a B-Spline collocation method for approximating higher order linear boundary value problems while a quintic B-spline and sixtic B-spline collocation methods were developed by [13,14] for the treatment of $8th$ order boundary value problems. Similarly, a Haar wavelet collocation method for approximating $8th$ order boundary value problems was developed by [15]. Cubic spline collocation tau method for handling 4th order linear ordinary differential equations was constructed by [16]. The Chebyshev polynomial was utilized by [17] to develop a multiple perturbed collocation tau-method which was used for solving $4th - 6th$ order BVPs. Again, [2] applied the Taylor series polynomials as basis to form a standard collocation method and further developed a perturbed collocation method using Chebyshev polynomials as perturbation terms for approximating $4th$ order BVPs. All the numerical methods mentioned above provided accurate approximations although with different accuracies.

The ease of implementing collocation methods with polynomial basis which provide accurate results that are comparable with other numerical methods is the motivation of this work. Since few orthogonal polynomials have been applied as trial functions to develop higher order collocation methods for solving BVPs, the Laguerre polynomial of degree $N = 10$ is utilized as basis function to construct a collocation method which is implemented on 2mth higher order BVPs, $2 \leq m \leq 5$. The existence and uniqueness of higher order BVPs are not considered in this work, however, this subject matter is comprehensively presented in [18] and [19].

The Laguerre collocation method is presented in Section 2, while the implementation is done in Section 3. Finally, Section 4 deals with the discussion and conclusion.

2. Methods

2.1. Laguerre polynomials

Laguerre polynomials are solutions of the Laguerre differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0, \tag{1}$$

obtained via series solution using the Frobenius method at the centre $x_0 = 0$. According to [20], Laguerre polynomials can be generated by the formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) = \frac{1}{n!}\left(\frac{d}{dx} - 1\right)^n x^n, \tag{2}$$

which satisfies the following recursive relationship

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x), \tag{3}$$

and

$$xL_n^{'}(x) = nL_n(x) - nL_{n-1}(x). \tag{4}$$

The first two terms $L_0(x)$ and $L_1(x)$ can be generated from (2) while the rest of terms can be obtained from (3) or (4). The first eleven terms which are polynomials of various degrees generated as explained above are given below

$$\left.\begin{aligned}
L_0(x) &= 1, \qquad L_1(x) = 1 - x, \qquad L_2(x) = 1 - 2x + \frac{1}{2}x^2,\\
L_3(x) &= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3,\\
L_4(x) &= 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4,\\
L_5(x) &= 1 - 5x + 5x^2 - \frac{5}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{120}x^5,\\
L_6(x) &= 1 - 6x + \frac{15}{2}x^2 - \frac{10}{3}x^3 + \frac{5}{8}x^4 - \frac{1}{20}x^5 + \frac{1}{720}x^6,\\
L_7(x) &= 1 - 7x + \frac{21}{2}x^2 - \frac{35}{6}x^3 + \frac{35}{24}x^4 - \frac{7}{40}x^5 + \frac{7}{720}x^6 - \frac{1}{5040}x^7,\\
L_8(x) &= 1 - 8x + 14x^2 - \frac{28}{3}x^3 + \frac{35}{12}x^4 - \frac{7}{40}x^5 + \frac{7}{180}x^6 - \frac{1}{630}x^7 + \frac{1}{40320}x^8,\\
L_9(x) &= 1 - 9x + 18x^2 - 14x^3\frac{21}{4}x^4 - \frac{21}{20}x^5 + \frac{7}{60}x^6 + \frac{1}{140}x^7 + \frac{1}{4480}x^8 - \frac{1}{362880}x^9,\\
L_{10}(x) &= 1 - 10x + \frac{45}{2}x^2 - 20x^3 + \frac{35}{4}x^4 - \frac{21}{10}x^5 + \frac{7}{24}x^6 - \frac{1}{42}x^7 + \frac{1}{896}x^8 - \frac{1}{36288}x^9 + \frac{1}{3628800}x^{10}.
\end{aligned}\right\} \tag{5}$$

2.2. Higher order linear boundary value problems

Consider the nth order ordinary differential equation

$$y^n = f\left(x, y, y', ..., y^{(n-1)}\right) \tag{6}$$

defined on the interval $a \leq x \leq b$ with the boundary conditions

$$\left.\begin{array}{rl} y(a) = A_0 & y(b) = B_0 \\ y^{'}(a) = A_1 & y^{'}(b) = B_1 \\ \vdots & \vdots \\ y^{(n-1)}(a) = A_{(n-1)} & y^{(n-1)}(b) = B_{(n-1)} \end{array}\right\} \tag{7}$$

The linear form of (6) is given as

$$p_n(x)y^{(n)}(x) + p_{(n-1)}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = g(x), \tag{8}$$

where $p_n(x), \ldots, p_0(x)$ are coefficients of the unknown function and its derivatives which may either be a constant or function of x, and similarly is $g(x)$.

This work seeks to obtain the approximate solution to (8) and the boundary conditions as given in (7). However, the boundary value problems considered here have the same order n with the number of boundary conditions k.

2.3. Derivation of the Laguerre collocation method

We assume that (8) and (7) can be approximated with a linear combination of the Laguerre polynomials provided in (5) as given below

$$y(x) = a_0L_0(x) + a_1L_1(x) + \cdots + a_NL_N(x), \tag{9}$$

where N is the degree of the polynomial, and $a_0, a_1, \ldots, a_N$ are constants to be determined. Equation (9) can be written in more compact sigma notation as

$$y(x) = \sum_{j=0}^{N} a_jL_j(x). \tag{10}$$

Equation (10) is differentiated n number of times corresponding to the order of the boundary value problem given and thereafter substituted in Equation (8) to get

$$p_n(x)\sum_{j=0}^{(N)} a_jL_j^{(n)}(x) + p_{n-1}(x)\sum_{j=0}^{N} a_jL_j^{(n-1)}(x) + \cdots + p_1(x)\sum_{j=0}^{N} a_jL_j'(x) + p_0(x)\sum_{j=0}^{N} a_jL_j(x) = g(x). \tag{11}$$

Each term of Equation (11) is expanded and the like coefficients $a_j, j = 0, 1, \ldots, N$ are collected resulting to

$$\sum_{j=0}^{N} a_j\left(p_j^{\star}(x)Q(x)\right) = g(x). \tag{12}$$

$Q_j(x)$ are polynomials of various degrees while $p_j^*(x)$ are coefficients which may be constants or functions of x which may not necessarily be polynomials.

2.4. Generating $N + 1$ systems of linear equations

Since the boundary value problems considered in this work have the same order n with the number of boundary conditions k, to solve for a_j, $j = 0, 1, \ldots, N, k$ number of equations are generated using the boundary conditions, $\frac{k}{2}$ each at the lower and upper boundaries respectively. The remaining $N-k+1$ equations are generated at the collocation points using Equation (12).

2.4.1. Generating k systems of linear equations using boundary conditions

The $k=2,,n$ equations generated using the boundary conditions are given as The $k=2,\ldots,n$ equations generated using the boundary conditions are given as

$$\left.\begin{array}{ll} \sum_{j=0}^{N} a_j L_j(a) = A_0 & \sum_{j=0}^{N} a_j L_j(b) = B_0 \\ \sum_{j=0}^{N} a_j L_j'(a) = A_1 & \sum_{j=0}^{N} a_j L_j'(b) = B_1 \\ \vdots & \vdots \\ \sum_{j=0}^{N} a_j L_j^{(n-1)}(a) = A_{n-1} & \sum_{j=0}^{N} a_j L_j^{(n-1)}(b) = B_{n-1} \end{array}\right\} \tag{13}$$

2.4.2. Generating $N-k+1$ systems of linear equations at the collocation points

First, we state the equation which is used to generate the collocation points

$$x_i = a + \frac{(b-a)i}{N-(k-2)}, i = 1,2,\ldots,N-(k-1). \tag{14}$$

Equation (14) is used to get the various collocation points which are substituted into Equation (12) to get the remaining $N-k+1$ equations. Thus Equation (12) can be recast as

$$\sum_{j=0}^{N} a_j \left(p_j^{\star}(x_i)Q(x_i)\right) = g(x_i), \tag{15}$$

for $i=1,2,\ldots,N-(k-1)$ and $a_j, j=0,1,,N$.

2.4.3. Representing the system of $N+1$ equations in matrix form

Altogether, Equations (13) and (15) give $N+1$ system of equations which can be written in matrix form

$$\begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,N+1} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,N+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n+1,1} & c_{n+1,2} & \cdots & c_{n+1,n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_N \end{pmatrix}, \tag{16}$$

where

$$c_{ij} = \begin{cases} \sum_{j=0}^{N} a_j L_j^{*}(a), \\ \sum_{j=0}^{N} a_j \left(p_j^{\star}(x_i)Q(x_i)\right) \\ \sum_{j=0}^{N} a_j L_j^{*}(b) \end{cases} \tag{17}$$

and

$$d_j = \begin{cases} A^{\star} \\ g(x_i) \\ B^{\star} \end{cases} \tag{18}$$

where $\sum_{j=0}^{N} a_j L_j^*(a)$, $\sum_{j=0}^{N} a_j L_j^*(b)$ and $A^\star$, $B^\star$ are the left and right hand side of the systems of k equations generated from the boundary conditions, while $\sum_{j=0}^{N} a_j \left(p_j^\star(x_i)Q(x_i)\right)$ and $g(x_i)$ is the left and right hand side of Equation (15) used to get the remaining $N-K+1$ system of equations.

3. Results

The Laguerre collocation method developed in Section 2 is applied to approximate some higher order linear boundary value problems. The approximate solutions inform of series solutions are displayed in Tables and compared to the analytical solution at some selected mesh points, while the accuracy is measured using absolute errors. Suppose x represents an independent variable and y the dependent variable, the approximate solution is denoted by y_n, the analytical solution by $y(x_n)$ and an absolute error at a mesh point by $e_n = |y_n - y(x_n)|$ in this work.

Example 1. Consider the $4th$ order boundary value problem $y^{(4)} = y + 4\exp(x), \quad 0 \leq x \leq 1$; with the boundary conditions $y(0) = 1,\ y(1) = 2\exp(1),\ \ y'(0) = 1,\ y'(1) = 3\exp(1)$ and the analytical solution is given by $y(x) = (1+x)exp(x)$. The results at the mesh points are given in Table 1.

Table 1. Approximate solution, analytical solution and absolute errors for Example 1

n	x_n	y_n	$y(x)$	$\lvert y_n - y(x_n)\rvert$
0	0	0.9999999995995449000	1.0000000000000000000	4.0046×10^{-10}
1	0.1	1.2156880096812333908	1.2156880098832123873	2.0198×10^{-10}
2	0.2	1.4656833097351525863	1.4656833097922038007	5.7051×10^{-11}
3	0.3	1.7548164498926186474	1.7548164498488040352	4.3815×10^{-11}
4	0.4	2.0885545768060973480	2.0885545766977784449	1.0832×10^{-10}
5	0.5	2.4730819061942987293	2.4730819060501922202	1.4411×10^{-10}
6	0.6	2.9153900807828490922	2.9153900806248143598	1.5803×10^{-10}
7	0.7	3.4233796028562222864	3.4233796026998100867	1.5641×10^{-10}
8	0.8	4.0059736714316378601	4.0059736712864416883	1.4519×10^{-10}
9	0.9	4.6732459113274837527	4.6732459111982043612	1.2928×10^{-10}
10	1.0	5.4365636570322238390	5.4365636569180904708	1.1413×10^{-10}

Example 2. Consider the $6th$ order boundary value problem $y^{(6)} = y - 6\exp(x), \quad 0 \leq x \leq 1$; with the boundary conditions $y(0) = 1,\ y(1) = 0,\ y^{(2)}(0) = -1,\ \ y^{(2)}(1) = -2\exp(1),\ \ y^{(4)}(0) = -3,\ y^{(2)}(1) = -3\exp(1)$, and the analytical solution is given by $y(x) = (1-x)exp(x)$. The results at the mesh points are given in Table 2.

Table 2. Approximate solution, analytical solution and absolute errors for Example 2

n	x_n	y_n	$y(x)$	$\lvert y_n - y(x_n)\rvert$
0	0	1.0000000000001685000	1.0000000000000000000	1.6850×10^{-13}
1	0.1	0.9946538258967874881	0.99465382626808286232	3.7130×10^{-10}
2	0.2	0.9771222054592636538	0.97712220652813586712	1.0689×10^{-9}
3	0.3	0.9449011631547264200	0.94490116530320217280	2.1485×10^{-9}
4	0.4	0.8950948151452182868	0.89509481858476219068	3.4395×10^{-9}
5	0.5	0.8243606307092493746	0.82436063535006407340	4.6408×10^{-9}
6	0.6	0.7288475147567316736	0.72884752015620358996	5.3995×10^{-9}
7	0.7	0.60412580684272018001	0.60412581224114295648	5.3984×10^{-9}
8	0.8	0.4451081812551064650	0.44510818569849352092	4.4434×10^{-9}
9	0.9	0.2459603085739087748	0.24596031111569496638	2.5418×10^{-9}
10	1.0	-3.25690×10^{-14}	0	3.2570×10^{-14}

Example 3. Consider the $8th$ order boundary value problem $y^{(8)} - y = 8\exp(x), \quad 0 \leq x \leq 1$; with the boundary conditions $y(0) = 1,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = -2,\ \ y^{(2)}(0) = -1,\ y^{(2)}(1) = -2\exp(1),$

$y^{(3)}(0) = -2$, $y^{(3)}(1) = -3\exp(1)$, and the analytical solution is given by $y(x) = (1-x)\exp(x)$. The results at the mesh points are given in Table 3.

Table 3. Approximate Solution, Analytical Solution and Absolute Errors for Example 3

n	x_n	y_n	$y(x)$	$\lvert y_n - y(x_n)\rvert$
0	0	0.999999999999919520	1.00000000000000000000	8.0480×10^{-14}
1	0.1	0.9946538262678937601	0.99465382626808286232	1.8910×10^{-13}
2	0.2	0.9771222065321117607	0.97712220652813586712	3.7959×10^{-12}
3	0.3	0.9449011653172361413	0.94490116530320217280	1.4034×10^{-11}
4	0.4	0.8950948185965164628	0.89509481858476219068	1.1754×10^{-11}
5	0.5	0.8243606353404964961	0.82436063535006407340	9.5676×10^{-12}
6	0.6	0.7288475201290590871	0.72884752015620358996	2.7415×10^{-11}
7	0.7	0.60412581221979442386	0.60412581224114295648	2.1348×10^{-11}
8	0.8	0.4451081856932311579	0.44510818569849352092	2.2624×10^{-12}
9	0.9	0.2459603111159681689	0.24596031111569496638	2.7320×10^{-13}
10	1.0	4.78512×10^{-14}	0	4.7851×10^{-14}

Example 4. Consider the $10th$ order boundary value problem $y^{(10)} = -(1-x)\sin(x) + 10\cos(x)$, $0 \le x \le 1$; with the boundary conditions $y(0) = 1$, $y(1) = 0$, $y^{(2)}(0) = 2$, $y^{(2)}(1) = 2\cos(1)$, $y^{(4)}(0) = -4$, $y^{(4)}(1) = -4\cos(1)$, $y^{(6)}(0) = 6$, $y^{(6)}(1) = 6\cos(1)$, $y^{(8)}(0) = -8$, $y^{(8)}(1) = -8\cos(1)$ and the analytical solution is given by $y(x) = (x-1)\sin(x)$. The results at the mesh points are given in Table 4.

Table 4. Approximate solution, analytical solution and absolute errors for Example 4

n	x_n	y_n	$y(x)$	$\lvert y_n - y(x_n)\rvert$
0	0	6.8300×10^{-17}	0	6.8300×10^{-17}
1	0.1	−0.0898511265815970128	−0.089850074982145337076	1.0516×10^{-6}
2	0.2	−0.1589374674789321263	−0.15893546463604897237	2.0028×10^{-6}
3	0.3	−0.2068669068902013555	−0.20686414466293770258	2.7622×10^{-6}
4	0.4	−0.2336542608365488710	−0.23365100538519029500	3.2555×10^{-6}
5	0.5	−0.2397162019696233407	−0.23971276930210150014	3.4327×10^{-6}
6	0.6	−0.2258602632950288019	−0.22585698935801414288	3.2739×10^{-6}
7	0.7	−0.19326809832291728014	−0.19326530617130731610	2.7922×10^{-6}
8	0.8	−0.1434732509647720103	−0.14347121817990455233	2.0328×10^{-6}
9	0.9	−0.0783337610764255054	−0.078332690962748338846	$2.1.070 \times 10^{-6}$
10	1.0	-4.8700×10^{-17}	0	4.8700×10^{-17}

4. Discussion and Conclusion

4.1. Discussion

In this section, the numerical experiments carried out with our proposed collocation method as presented in Section 3 are discussed. MAPLE 17 is used to implement all the problems. To reduce round-off errors, the numerical approximations were rounded up to 20 digits.

The Laguerre polynomial of degree 10 was used to develop an orthogonal collocation method for solving higher order boundary value problems in ordinary differential equations. Four test problems on $4th$, $6th$, $8th$ and $10th$ order boundary value problems were used to verify the efficiency and accuracy of the proposed method via absolute errors. The numerical results are displayed in Tables 1-4. The results from Tables 1-3 which are BVPs of order 4, 6 and 8 respectively are highly accurate, while the result in Table 4 which is a BVP of order 10 is fairly accurate when compared to the other problems. In general, our proposed collocation method provides an accurate numerical method for approximating higher order BVPs. However, we observed from Tables 1-4 that the accuracy of the numerical results increased at the boundaries as the order and number of boundary conditions also increased. On the other hand, the accuracy at the interior mesh points were less accurate to that at the boundary points as the order and number of boundary conditions increased. This may be as a result of the increase in the number of boundary conditions and a corresponding decrease in the number of collocation points. This may be the reason for the poor accuracy of Example 4 which has only one collocation point and 10 boundary conditions. To develop a collocation method that may handle such higher order BVPs, it is advisable to consider many basis terms in order to get higher order polynomials so as to have many collocation points which may be equal or more than the equations obtained at the boundaries.

4.2. Conclusion

The Laguerre polynomial which is an orthogonal polynomial was used as a basis function to develop a collocation method. The proposed method was easier to develop and implement as compared to other functions which are used as basis for developing other collocation methods. The method is also accurate and comparable to many other collocation methods in literature. The collocation method can be extended to solve higher order BVPs by considering higher order Laguerre polynomials. Other orthogonal polynomials may similarly be used to develop collocation methods for handling higher order BVPs.

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