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Čebyšev inequalities for co-ordinated QC-convex and (s, QC)-convex

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Abstract: In this paper, we establish some new Čebyšev type inequalities for functions whose modulus of the mixed derivatives are co-ordinated quasi-convex and α-quasi-convex and s-quasi-convex functions.

Keywords: Čebyšev inequalities, quasi-convexity, (s, QC)-convexity, (α, QC)-convexity.

1. Introduction

In 1882, Čebyšev [1] gave the following inequality

$$|T(f,g)| \leq \tfrac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1}$$

where $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous function, whose first derivatives f' and $g'$are bounded and

$$T(f,g) = \tfrac{1}{b-a}\int_a^b f(x)\, g(x)\, dx - \left(\tfrac{1}{b-a}\int_a^b f(x)\, dx\right)\left(\tfrac{1}{b-a}\int_a^b g(x)\, dx\right), \tag{2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess\sup\limits_{t\in[a,b]} |f(t)|$.

During the past few years, many researchers have given considerable attention to the inequality (1). Various generalizations, extensions and variants have been appeared in the literature [2–6].

Recently, Guezane-Lakoud and Aissaoui [2] gave the analogue of the functional (2) for functions of two variables and established the following Čebyšev type inequalities for functions whose mixed derivatives are bounded as follows;

$$|T(f,g)| \leq \tfrac{49}{3600}k^2 \|f_{\lambda\alpha}\|_\infty \|g_{\lambda\alpha}\|_\infty, \tag{3}$$

and

$$|T(f,g)| \leq \frac{1}{8k^2}\int_a^b\int_c^d \left[\left(|g(x,y)|\,\|f_{\lambda\alpha}\|_\infty + |f(x,y)|\,\|g_{\lambda\alpha}\|_\infty\right)\; \left[\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right]\right] dydx, \tag{4}$$

where

$$\begin{aligned} T(f,g) =& \tfrac{1}{k}\int_a^b\int_c^d f(x,y)\, g(x,y)\, dydx - \tfrac{d-c}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)\, dt\right) dydx \\ &- \tfrac{b-a}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,v)\, dv\right) dydx + \tfrac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)\, dydx\right)\left(\int_a^b\int_c^d g(t,v)\, dvdt\right). \end{aligned} \tag{5}$$

Motivated by the existing results, in this paper we establish some new Čebyšev type inequalities for functions whose mixed derivatives are co-ordinates quasi-convex and co-ordinates (α, QC) and (s, QC)-convex.

2. Preliminaries

Throughout this paper, we denote by Δ, the bidimensional interval in $[0,\infty)^2$, $\Delta =: [a,b] \times [c,d]$ with $a < b$ and $c < d$, $k =: (b-a)(d-c)$ and $\frac{\partial^2 f}{\partial\lambda\partial w}$ by $f_{\lambda w}$.

Definition 1. [7] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ if

$$f(\lambda x + (1-\lambda)t, wy + (1-w)v) \leq \lambda w f(x,y) + \lambda(1-w) f(x,v) + (1-\lambda) w f(t,y) + (1-\lambda)(1-w) f(t,v)$$

holds for all $\lambda, w \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Definition 2. [8] A function $f : \Delta \to \mathbb{R}$ is said to be quasi-convex on the co-ordinates on Δ if

$$f(\lambda x + (1-\lambda)t, wy + (1-w)v) \leq \max\{f(x,y) + f(x,v) + f(t,y) + f(t,v)\}$$

holds for all $\lambda, w \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Definition 3. [9] For some $\alpha \in (0,1]$, a function $f : \Delta \to \mathbb{R}$ is said to be (α, QC)-convex on the co-ordinates on Δ, if

$$f(\lambda x + (1-\lambda)t, wy + (1-w)v) \leq \lambda^\alpha \max\{f(x,y) + f(x,v)\} + (1-\lambda^\alpha)\max\{f(t,y) + f(t,v)\}$$

holds for all $\lambda, w \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Definition 4. [10] For some $s \in [-1,1]$, a function $f : \Delta \to [0,\infty)$ is said to be (s, QC)-convex on co-ordinates on Δ, if

$$f(\lambda x + (1-\lambda)t, wy + (1-w)v) \leq \lambda^s \max\{f(x,y) + f(x,v)\} + (1-\lambda)^s \max\{f(t,y) + f(t,v)\}$$

holds for all $\lambda \in (0,1)$, $w \in [0,1]$ and $(x,y), (x,v), (t,y), (t,v) \in \Delta$.

Lemma 1. [11] *Let $f : \Delta \to \mathbb{R}$ be a partial differentiable mapping on Δ in $\mathbb{R}^2$. If $f_{\lambda w} \in L_1(\Delta)$ then for any $(x,y) \in \Delta$, we have the equality;*

$$\begin{aligned} f(x,y) =& \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{d-c}\int_c^d f(x,v)dv - \frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt + \frac{1}{k}\int_a^b\int_c^d (x-t)(y-v) \\ &\times \left(\int_0^1\int_0^1 f_{\lambda w}(\lambda x + (1-\lambda)t, wy - (1-w)v)\, dwd\lambda\right) dvdt. \end{aligned} \tag{6}$$

3. Main result

Theorem 1. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda w}$ and $g_{\lambda w}$ are integrable on Δ. If $|f_{\lambda w}|$ and $|g_{\lambda w}|$ are co-ordinated quasi-convex on Δ, then*

$$|T(f,g)| \leq \frac{49}{3600} MNk^2, \tag{7}$$

where $T(f,g)$ is defined as in (5), $M = \max\limits_{x,t\in[a,b],y,v\in[c,d]} [|f_{\lambda w}(x,y)| + |f_{\lambda w}(x,v)| + |f_{\lambda w}(t,y)| + |f_{\lambda w}(t,v)|]$, and $N = \max\limits_{x,t\in[a,b],y,v\in[c,d]} [|g_{\lambda w}(x,y)| + |g_{\lambda w}(x,v)| + |g_{\lambda w}(t,y)| + |g_{\lambda w}(t,v)|]$, and $k = (b-a)(d-c)$.

Proof. From Lemma 1, we have

$$f(x,y)-\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{d-c}\int_c^d f(x,v)dv+\frac{1}{k}\int_a^b\int_c^d f(t,v)dvdt$$
$$=\frac{1}{k}\int_a^b\int_c^d (x-t)\,(y-v)\left(\int_0^1\int_0^1 f_{\lambda w}\left(\lambda x+(1-\lambda)t,wy-(1-w)v\right)d\alpha d\lambda\right)dvdt, \tag{8}$$

and

$$g(x,y)-\frac{1}{b-a}\int_a^b g(t,y)dt-\frac{1}{d-c}\int_c^d g(x,v)dv+\frac{1}{k}\int_a^b\int_c^d g(t,v)dvdt$$
$$=\frac{1}{k}\int_a^b\int_c^d (x-t)\,(y-v)\left(\int_0^1\int_0^1 g_{\lambda w}\left(\lambda x+(1-\lambda)t,wy-(1-w)v\right)dwd\lambda\right)dvdt. \tag{9}$$

Multiplying (8) by (9), and then integrating the resulting equality with respect to x and y over Δ, using modulus and Fubini's Theorem, and multiplying the result by $\frac{1}{k}$, we get

$$|T(f,g)|\leq\frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d|x-t|\,|y-v|\times\left(\int_0^1\int_0^1\left|f_{\lambda w}\left(\lambda x+(1-\lambda)t,wy-(1-w)v\right)\right|dwd\lambda\right)dvdt\right]$$
$$\times\left[\int_a^b\int_c^d|x-t|\,|y-v|\times\left(\int_0^1\int_0^1 g_{\lambda w}\left|(\lambda x+(1-\lambda)t,wy-(1-w)v)\right|dwd\lambda\right)\right]dvdt\Bigg]dydx. \tag{10}$$

Since $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are co-ordinated quasi-convex, we deduce

$$|T(f,g)|\leq\frac{1}{k^3}MN\int_a^b\int_c^d\left(\int_a^b\int_c^d|x-t|\,|y-v|\,dvdt\right)^2dydx=\frac{49}{3600}k^2MN, \tag{11}$$

where we have used the fact that

$$\int_a^b\int_c^d\left(\int_a^b\int_c^d|x-t|\,|y-v|\,dvdt\right)^2dydx=\frac{49}{3600}k^5. \tag{12}$$

The proof is completed. □

Theorem 2. *Under the assumptions of Theorem 1, we have*

$$|T(f,g)|\leq\frac{1}{8k^2}\left[\int_a^b\int_c^d\left[M\,|g(x,y)|+N\,|f(x,y)|\right]\left[(x-a)^2+(b-x)^2\right]\times\left[(y-c)^2+(d-y)^2\right]\right]dydx, \tag{13}$$

where $T(f,g)$ is defined as in (5), M, N, and k are as in Theorem 1.

Proof. From Lemma 1, (8) and (9) are valid. Let $G(x,y)=\frac{1}{2k}g(x,y)$ and $F(x,y)=\frac{1}{2k}f(x,y)$. Multiplying $G(x,y)$ by $F(x,y)$, then integrating the resultant equalities with respect to x and y over Δ, and by using the modulus, we get

$$
\begin{aligned}
|T(f,g)| \leq & \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\right.\\
& \times \left.\left(\int_0^1\int_0^1 |f_{\lambda w}(\lambda x+(1-\lambda)t, wy-(1-w)v)|\,dwd\lambda\right) dvdt\right] dydx + \int_a^b\int_c^d |f(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
& \times \left.\left.\left(\int_0^1\int_0^1 |g_{\lambda w}(\lambda x+(1-\lambda)t, wy-(1-w)v)|\,dwd\lambda\right) dvdt\right] dydx. \qquad (14)
\end{aligned}
$$

Since $|f_{\lambda w}|$ and $|g_{\lambda w}|$ are co-ordinated quasi-convex, (14) implies

$$
\begin{aligned}
|T(f,g)| \leq & \frac{1}{2k^2}\left[\int_a^b\int_c^d M\,|g(x,y)| \left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right) dydx\right.\\
& \left.+\int_a^b\int_c^d N\,|f(x,y)| \left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)\right] dydx\\
= & \frac{1}{2k^2}\int_a^b\int_c^d (M\,|g(x,y)|+N\,|f(x,y)|) \left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right) dydx. \qquad (15)
\end{aligned}
$$

By a simple computation, we easily obtain

$$
\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt = \frac{1}{4}\left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right]. \qquad (16)
$$

Substituting (16) in (15), we get the desired result. □

Theorem 3. *Let $f,g:\Delta\to\mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda w}$ and $g_{\lambda w}$ are integrable on Δ. If $|f_{\lambda w}|$ and $|g_{\lambda w}|$ are co-ordinated α-quasi-convex on Δ, for some $\alpha\in(0,1]$, then*

$$
|T(f,g)| \leq \frac{49}{3600}MNk^2, \qquad (17)
$$

where $T(f,g)$ is defined as in (5), M, N, and k are as in Theorem 1.

Proof. Clearly the inequalities (8)-(10) are valid, using the co-ordinated α-quasi-convexity of $|f_{\lambda w}|$ and $|g_{\lambda w}|$, (10) gives

$$
\begin{aligned}
|T(f,g)| \leq & \frac{1}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-v| \int_0^1\int_0^1 [\lambda^\alpha \max\{|f_{\lambda w}(x,y)|+|f_{\lambda w}(x,v)|\}\right.\\
& + \left.(1-\lambda^\alpha)\max\{|f_{\lambda w}(t,y)|+|f_{\lambda w}(t,v)|\}]\,dwd\lambda)\,dvdt\right]\\
& \times \left[\int_a^b\int_c^d |x-t|\,|y-v| \int_0^1\int_0^1 [\lambda^\alpha \max\{|g_{\lambda w}(x,y)|+|g_{\lambda w}(x,v)|\}\right.\\
& + \left.(1-\lambda^\alpha)\max\{|g_{\lambda w}(t,y)|+|g_{\lambda w}(t,v)|\}]\,dwd\lambda]\,dvdt\right] dydx\\
= & \frac{1}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-v| \left[\max\{|f_{\lambda w}(x,y)|+|f_{\lambda w}(x,v)|\}\int_0^1\int_0^1 \lambda^\alpha dwd\lambda\right.\right.
\end{aligned}
$$

$$+ \max\{|f_{\lambda w}(t,y)| + |f_{\lambda w}(t,v)|\} \int_0^1\int_0^1 (1-\lambda^\alpha)\, dwd\lambda \Bigg] dvdt \Bigg]$$

$$\times \left[\int_a^b\int_c^d |x-t|\,|y-v| \Bigg[\max\{|g_{\lambda w}(x,y)| + |g_{\lambda w}(x,v)|\} \int_0^1\int_0^1 \lambda^\alpha dwd\lambda \right.$$

$$\left. + \max\{|g_{\lambda w}(t,y)| + |g_{\lambda w}(t,v)|\} \int_0^1\int_0^1 (1-\lambda^\alpha)\, dwd\lambda \Bigg] dvdt \right] dydx$$

$$\leq \frac{1}{k^3} \int_a^b\int_c^d \left[\left[\left(\int_a^b\int_c^d |x-t|\,|y-v| \left(\frac{1}{\alpha+1} + 1 - \frac{1}{\alpha+1} \right) Mdvdt \right) \right] \right.$$

$$\left. \times \left[\left(\int_a^b\int_c^d |x-t|\,|y-v| \left(\frac{1}{\alpha+1} + 1 - \frac{1}{\alpha+1} \right) Ndvdt \right) \right] \right] dydx$$

$$= \frac{MN}{k^3} \int_a^b\int_c^d \left(\int_a^b\int_c^d |x-t|\,|y-v| \right)^2 dydx. \tag{18}$$

Using (12) in (18), we obtain the desired result. □

Theorem 4. *Under the assumptions of Theorem 3, we have*

$$|T(f,g)| \leq \frac{1}{8k^2} \left[\int_a^b\int_c^d (M\,|g(x,y)| + N\,|f(x,y)|) \times \left[(x-a)^2 + (b-x)^2\right] \left[(y-c)^2 + (d-y)^2\right] dydx, \right. \tag{19}$$

where $T(f,g)$ is defined as in (5) and $M, N,$ and k are as in Theorem 3.

Proof. By the same argument given in Theorem 2, we easly obtain the inequality (14), using the α-quasi-convexity on the co-ordinates of $|f_{\lambda w}|$ and $|g_{\lambda w}|$, we get

$$|T(f,g)| \leq \frac{1}{2k^2} \left[\int_a^b\int_c^d |g(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-v| \times \left(M\int_0^1\int_0^1 \lambda^\alpha dwd\lambda + M\int_0^1\int_0^1 (1-\lambda^\alpha)\, dwd\lambda \right) dvdt \right] dydx \right.$$

$$+ \int_a^b\int_c^d |f(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-v| \times \left(N\int_0^1\int_0^1 \lambda^\alpha dwd\lambda + N\int_0^1\int_0^1 (1-\lambda^\alpha)\, dwd\lambda \right) dvdt \right] dydx.$$

$$= \frac{1}{2k^2} \int_a^b\int_c^d \left[(M\,|g(x,y)| + N\,|f(x,y)|) \int_a^b\int_c^d |x-t|\,|y-v|\, dvdt \right] dydx. \tag{20}$$

Substituting (16) in (20), we get the desired result. □

Theorem 5. *Let $f, g : \Delta \to \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda w}$ and $g_{\lambda w}$ are integrable on Δ, and let $s \in (-1, 1]$ fixed. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are co-ordinated s-quasi-convex on Δ, then*

$$|T(f,g)| \leq \frac{49}{900(s+1)^2} MNk^2, \tag{21}$$

where $T(f,g)$ is defined as in (5) and $M, N,$ and k are as in Theorem 1.

Proof. Clearly inequalities (8)-(10) are satisfied. Using second definition of the co-ordinated s-quasi-convex of $|f_{\lambda w}|$ and $|g_{\lambda w}|$, (10) gives;

$$\begin{aligned}
|T(f,g)| \leq & \frac{1}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-v| \int_0^1\int_0^1 [\lambda^s \max\{|f_{\lambda w}(x,y)|+|f_{\lambda w}(x,v)|\}\right. \\
& + \left.(1-\lambda)^s \max\{|f_{\lambda w}(t,y)|+|f_{\lambda w}(t,v)|\}]\, dwd\lambda)\, dvdt\right] \\
& \times \left[\int_a^b\int_c^d |x-t|\,|y-v| \int_0^1\int_0^1 [\lambda^s \max\{|g_{\lambda w}(x,y)|+|g_{\lambda w}(x,v)|\}\right. \\
& + \left.(1-\lambda)^s \max\{|g_{\lambda w}(t,y)|+|g_{\lambda w}(t,v)|\}]\, dwd\lambda]\, dvdt\right] dydx \\
= & \frac{1}{k^3}\int_a^b\int_c^d \left[\int_a^b\int_c^d |x-t|\,|y-v| \left[\max\{|f_{\lambda w}(x,y)|+|f_{\lambda w}(x,v)|\}\int_0^1\int_0^1 \lambda^s dwd\lambda\right.\right. \\
& + \left.\left.\max\{|f_{\lambda w}(t,y)|+|f_{\lambda w}(t,v)|\}\int_0^1\int_0^1 (1-\lambda)^s\, dwd\lambda\right] dvdt\right] \\
& \times \left[\int_a^b\int_c^d |x-t|\,|y-v| \left[\max\{|g_{\lambda w}(x,y)|+|g_{\lambda w}(x,v)|\}\int_0^1\int_0^1 \lambda^s dwd\lambda\right.\right. \\
& + \left.\left.\max\{|g_{\lambda w}(t,y)|+|g_{\lambda w}(t,v)|\}\int_0^1\int_0^1 (1-\lambda)^s\, dwd\lambda\right] dvdt\right] dydx \\
\leq & \frac{1}{k^3}\int_a^b\int_c^d \left[\left[\left(\int_a^b\int_c^d |x-t|\,|y-v| \left(\frac{M}{s+1}+\frac{M}{s+1}\right) dvdt\right)\right]\right. \\
& \times \left.\left[\left(\int_a^b\int_c^d |x-t|\,|y-v| \left(\frac{N}{s+1}+\frac{N}{s+1}\right) dvdt\right)\right]\right] dydx \\
= & \frac{4MN}{(s+1)^2k^3}\int_a^b\int_c^d \left(\int_a^b\int_c^d |x-t|\,|y-v|\right)^2 dydx. \qquad (22)
\end{aligned}$$

Substituting (12) in (22), we get the desired result. □

Theorem 6. *Under the assumptions of Theorem 5, we have*

$$|T(f,g)| \leq \frac{1}{4(s+1)k^2}\left[\int_a^b\int_c^d (M\,|g(x,y)|+N\,|f(x,y)|) \times \left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right] dydx, \right. \qquad (23)$$

where $T(f,g)$ is defined as in (5) and M, N, and k are as in Theorem 1.

Proof. By the same argument given in Theorem 2, we easily obtain the inequality (14), using the second definition of s-quasi-convexity on the co-ordinates of $|f_{\lambda w}|$ and $|g_{\lambda w}|$, we get

$$\begin{aligned}
|T(f,g)| \leq & \frac{1}{2k^2}\left[\int_a^b\int_c^d |g(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-v| \times \left(M\int_0^1\int_0^1 \lambda^s dwd\lambda + M\int_0^1\int_0^1 (1-\lambda)^s\, dwd\lambda\right) dvdt\right] dydx\right. \\
& + \left.\int_a^b\int_c^d |f(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-v| \times \left(N\int_0^1\int_0^1 \lambda^s dwd\lambda + N\int_0^1\int_0^1 (1-\lambda)^s\, dwd\lambda\right) dvdt\right] dydx.\right. \\
= & \frac{1}{(s+1)k^2}\int_a^b\int_c^d \left[(M\,|g(x,y)|+N\,|f(x,y)|)\int_a^b\int_c^d |x-t|\,|y-v|\, dvdt\right] dydx. \qquad (24)
\end{aligned}$$

Substituting (16) in (24), we get the desired result. □

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References

[1] Čebyšev, P. L. (1882). Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. *In Proc. Math. Soc. Charkov, 2*, 93-98.

[2] Guezane-Lakoud, A., & Aissaoui, F. (2011). New Čebyšev type inequalities for double integrals. *Journal of Mathematical Inequalities, 5*(4), 453-462.

[3] Meftah, B., & Boukerrioua, K. (2015). New Čebyšev type inequalities for functions whose second derivatives are $(s_1, m_1) - (s_2, m_2)$ convex on the co-ordinates. *Theory and Applications of Mathematics & Computer Science, 5*(2), 116-125.

[4] Meftah, B., & Boukerrioua, K. (2015). Čebyšev inequalities whose second derivatives are $(s, r)-$convex on the co-ordinates. *Journal of Advanced Research in Applied Mathematics, 7*(3), 76-87.

[5] Meftah, B., & Boukerrioua, K. (2015). On some Čebyšev yupe inequalities for functions whose second derivative are $(h_1; h_2)$-convex on the co-ordinates. *Konuralp Journal of Mathematics, 3*(2), 77-88.

[6] Sarikaya, M. Z., Budak, H., & Yaldiz, H. (2014). Čebyšev type inequalities for co-ordinated convex functions. *Pure and Applied Mathematics Letters, 2*, 44-48.

[7] Dragomir, S. S. (2001). On the Hadamard's inequlality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese Journal of Mathematics, 5*(4), 775-788.

[8] Latif, M. A., Hussain, S., & Dragomir, S. S. (2012). Refinements of Hermite–Hadamard type inequalities for co-ordinated quasi–convex functions. *International Journal of Mathematical Archive, 3*(1), 161-171.

[9] Xi, B. Y., Sun, J., & Bai, S. P. (2015). On some Hermite-Hadamard type integral inequalities for co-ordinated (a, QC) and $(a, CJ)-$convex functions. *Tbilisi Mathematical Journal, 8*(2), 75-86.

[10] Wu, Y., & Qi, F. (2016). On some Hermite–Hadamard type inequalities for (s, QC)–convex functions. *SpringerPlus, 5*(1), 1-13.

[11] Sarikaya, M. Z. (2014). On the Hermite–Hadamard type inequalities for co-ordinated convex function via fractional integrals. *Integral Transforms and Special Functions, 25*(2), 134-147.