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A Unit Norm Conjecture for Some Real Quadratic Number Fields: A Preliminary Heuristic Investigation

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper we make a conjecture about the norm of the fundamental unit, N(e), of some real quadratic number fields that have the form $k = Q(\sqrt{(p_1.p_2)}$ where p_1 and p_2 are distinct primes such that $p_i = 2$ or $p_i \equiv 1 \bmod 4$, i = 1, 2. Our conjecture involves the case where the Kronecker symbol $(p_1/p_2) = 1$ and the biquadratic residue symbols $(p_1/p_2)_4 = (p_2/p_1)_4 = 1$, and is based upon Stevenhagen's conjecture that if $k = Q(\sqrt{(p_1.p_2)}$ is any real quadratic number field as above, then $P(N(e) = -1)) = 2/3$, i.e., the probability density that $N(e) = -1$ is 2/3. Given Stevenhagen's conjecture and some theoretical assumptions about the probability density of the Kronecker symbols and biquadratic residue symbols, we establish that if k is as above with $(p_1/p_2) = (p_1/p_2)_4 = (p_2/p_1)_4 = 1$, then $P(N(e) = -1)) = 1/3$, and we support our conjecture with some preliminary heuristic data.

Keywords: Negative Pell equation; unit norm conjecture; real quadratic number field; heuristic investigation; Kronecker symbol; biquadratic residue symbol; chi-square test.

1 Introduction

In this paper we make a conjecture about the norm of the fundamental unit, N(e), of some real quadratic number fields of the form $k = Q(\sqrt{(p_1.p_2)}$ where p_1 and p_2 are distinct primes such that $p_i = 2$ or $p_i \equiv 1 \bmod 4$, i = 1, 2. It is well known that if a prime congruent to 3 mod 4 divides the discriminant d_k of a real quadratic

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number field k then N(e) = 1, which is equivalent to the negative Pell equation $x^2 - d_k y^2 = -1$, x and y integers, not being solvable, and that if a prime congruent to 3 mod 4 does not divide d_k then N(e) = 1 or -1 [1]. In addition to this algebraic number theory unit norm application of the negative Pell equation, there are also a number of other applications, inclusive of Chebyshev polynomials and continued fractions [2]. In the above case where $k = Q(\sqrt{(p_1.p_2)}$ it is also well known that if the Kronecker symbol $(p_1/p_2) = -1$ then we have N(e) = -1, and that if $(p_1/p_2) = 1$ and the biquadratic residue symbols $(p_1/p_2)_4(p_2/p_1)_4 = -1$ then N(e) = 1[1]. Furthermore, it is known that if $(p_1/p_2) = 1$ and $(p_1/p_2)_4 = (p_2/p_1)_4 = -1$ then N(e) = -1 [3,4]. However, in the case where $(p_1/p_2) = 1$ and $(p_1/p_2)_4 = (p_2/p_1)_4 = 1$ we may have N(e) = 1 or -1 [3,4].

Stevenhagen [1] has made the conjecture that if $k = Q(\sqrt{(p_1.p_2)}$ is any real quadratic number field as above, then P(N(e) = 1)) = 2/3, i.e., the probability density that N(e) = -1 is 2/3. This conjecture by Stevenhagen is actually part of his deeper and more extensive conjecture which states that if one considers any real quadratic number field k such that its discriminant does not contain a prime congruent to 3 mod 4, then P(N(e) = -1)) ≈ .581 [1]). Recently, Knight and Xiao [5] claimed to prove this wider conjecture by Stevenhagen. However, their claim has been challenged by Chan et al. [6], as Chan et al. were not able to verify one of Knight and Xiao's equations that was needed to prove Stevenhagen's wider conjecture. The techniques that Chan et al. [6] utilized is based heavily upon analytic number theory, algebraic number theory, and probability theory, in particular upon previous work by Alexander Smith [7,8] with an emphasis on the properties of the Rédei symbol, as well as upon concepts and techniques involving 2-class groups, the Chebotarev Density Theorem, the Cauchy-Schwarz inequality, Markov chain modeling, the prime number theorem, Chernoff bounds, and the Artin map (see e.g., Alon & Spencer [9]; Fouvry and Kluners [10]; Stevenhagen [11]). Chan et al. [6] was able to prove that in regard to the wider Stevenhagen conjecture, P(N(e) = -1)) ≥ .538.

Our conjecture in this paper involves Stevenhagen's [1] more limited conjecture that P(N(e) = -1)) = 2/3 for real quadratic number fields of the form $k = Q(\sqrt{(p_1.p_2)}$ where p_1 and p_2 are distinct primes such that $p_i = 2$ or $p_i \equiv 1 \bmod 4$, i = 1, 2. Stevenhagen [1] included heuristic data along with some theoretical assumptions to formulate both his wider and more limited conjecture, but neither his heuristic data nor that of any previous publications we have seen in the literature test his limited conjecture in the form that we have given it in Lemma 1. In Lemma 1 we make the additional assumption that $(p_1/p_2) = (p_1/p_2)_4 = (p_2/p_1)_4 = 1$, and assuming Stevenhagen's conjecture and some theoretical assumptions about the Kronecker symbol and biquadratic residue symbols between p_1 and p_2 (see Assumptions 1 and 2), we establish that P(N(e) = -1)) = 1/3. We support our conjecture in Lemma 1, as well as our theoretical assumptions, with some preliminary heuristic data (see Results 1 and 2, and Table 1).

2 Conjecture that P(N(e) = -1)) = 1/3

We begin with the following theoretical assumptions:

Assumption 1: Let p_1 and p_2 be distinct primes such that $p_i = 2$ or $p_i \equiv 1 \bmod 4$, i = 1, 2. Then all Kronecker symbols and biquadratic residue symbols between the primes have an equal likelihood of occurring. Thus $P((p_1/p_2) = 1) = P((p_1/p_2) = -1) = .5$, and if $(p_1/p_2) = 1$ then $P((p_1/p_2)_4 = 1) = P((p_1/p_2)_4 = -1) = .5$, $P((p_1/p_2)_4.(p_2/p_1)_4 = 1) = P((p_1/p_2)_4.(p_2/p_1)_4 = -1) = .5$, and $P((p_1/p_2)_4 = (p_2/p_1)_4) = 1) = P((p_1/p_2)_4 = (p_2/p_1)_4) = -1) = .25$.

Remark 1: See Result 1 for some supportive heuristic data that if $k = Q(\sqrt{(p_1.p_2)}$ as above then $P((p_1/p_2) = 1) = P((p_1/p_2) = -1) = .5$, and that if $p_1 \equiv 1 \bmod 8$ and $p_2 = 2$ (which implies that $(p_1/p_2) = 1$) then $P((p_1/p_2)_4 = 1) = P((p_1/p_2)_4 = -1) = .5$. We leave it to the interested reader to check that heuristic data supports the general assumption that if $(p_1/p_2) = 1$) then $P((p_1/p_2)_4 = 1) = P((p_1/p_2)_4 = -1) = .5$ for $p_i = 2$ or $p_i \equiv 1 \bmod 4$, i = 1, 2. The remainder of the assumptions follow by the laws of conditional probability from this latter assumption [12].

Assumption 2: We assume Stevenhagen's conjecture [1] that if $k = Q(\sqrt{(p_1.p_2)}$ as above then P(N(e) = -1)) = 2/3.

Remark 2: See Result 1 for some supportive heuristic data for Stevenhagen's conjecture; i.e., that if $k = Q(\sqrt{(p_1.p_2)}$ as above then $P(N(e) = -1)) = 2/3$.

Given Assumptions 1 and 2, we are able to establish the following lemma:

Lemma 1: Let $k = Q(\sqrt{(p_1.p_2)}$ as above, $(p_1/p_2) = 1$, and $(p_1/p_2)_4 = (p_2/p_1)_4 = 1$. If Assumptions 1 and 2 hold then $P(N(e) = -1)) = 1/3$.

Proof: From Lemmermeyer [3] and Scholz [4] we know that if $(p_1/p_2) = 1$ and $(p_1/p_2)_4.(p_2/p_1)_4 = -1$ then $N(e) = 1$, and from Assumption 1 that $(p_1/p_2) = 1$ occurs 50% of the time. We also know from Lemmermeyer [3] and Scholz [4] that if $(p_1/p_2) = 1$ and $(p_1/p_2)_4 = (p_2/p_1)_4 = -1$ then $N(e)= -1$, and we have from Assumption 1 that if $(p_1/p_2) = 1$ then $(p_1/p_2)_4 = (p_2/p_1)_4 = -1$ occurs 25% of the time. We know that if $(p_1/p_2) = -1$ then $N(e) = -1$, and from Assumption 1 that $P((p_1/p_2) = -1) = .5$. Using Assumption 1 again, we thus have the following probability percentage contributions for $N(e) = -1$: 50% from $(p_1/p_2) = -1$, and from the laws of conditional probability [12], $(1/4).(1/2) = 1/8 = 12.5\%$ from $(p_1/p_2) = 1$ and $(p_1/p_2)_4 = (p_2/p_1)_4 = -1$. From Assumptions 1 and 2 we therefore conclude that if $(p_1/p_2) = (p_1/p_2)_4 = (p_2/p_1)_4 = 1$ then $P(N(e) = -1)) = 1/3$ since we have that $2/3 - (1/2 + 1/8) = 2/3 – 5/8 = 1/24$, $P((p_1/p_2) = (p_1/p_2)_4 = (p_2/p_1)_4 = 1) = 1/8$, and $1/24 = (1/3).(1/8)$.

3 Preliminary Heuristic Data That Supports Lemma 1

We utilized the Keith Matthews [13] website to calculate the norms of fundamental units of our real quadratic number fields. To assist in calculating our biquadratic residue symbols we used the following two well-known results [14].

Lemma 2 (Gauss): Let p be a prime congruent to 1 mod 8 with $p = a^2 + b^2$, b even.

Then $(2/p)_4 = (-1)^{(b/4)}$.

Lemma 3 (Burde): Let p and q be primes congruent to 1 mod 4 with $p = a^2 + b^2$ and $q = c^2 + d^2$ and b, d even, and assume that $(p/q) = 1$. Then $(p/q)_4(q/p)_4 = ((ac + bd)/p) = ((ac + bd)/q$.

We summarize our preliminary heuristic data as follows, where $k = Q(\sqrt{(p_1.p_2)}$ as above (with possibly a condition for the biquadratic residue symbols), N is the number of fields k such that $d_k < A$, and N_p denotes the number of fields k such that $d_k < A$ with the stated condition that involves either the Kronecker symbol or the unit norm.

Result 1: A: For $p_1 \equiv p_2 \equiv 1 \bmod 4$ and $d_k < A = 10{,}000$: $N = 452$; if $(p_1/p_2) = 1$ then $N_p = 214$ and $N_p/N = 214/452 \approx 47.3\%$; if $(p_1/p_2) = -1$ then $N_p = 238$ and $N_p/N = 238/452 \approx 52.7\%$.

B: For $p_1 = 2$ and $p_2 \equiv 1 \bmod 4$ (wlog) and $d_k < A = 25{,}000$: $N = 215$; if $(p_1/p_2) = 1$ (i.e., $p_2 \equiv 1 \bmod 8$) then $N_p = 105$ and $N_p/N = 105/215 \approx 48.8\%$; if $(p_1/p_2) = -1$ (i.e., $p_2 \equiv 5 \bmod 8$) then $N_p = 110$ and $N_p/N = 110/215 \approx 51.2\%$.

C: For $p_1 = 2$ and $p_2 \equiv 1 \bmod 8$ (wlog) and $d_k < A = 25{,}000$: $N = 105$; if $(p_2/p_1)_4 = 1$ (i.e., $p_2 \equiv 1 \bmod 16$), then $N_p = 53$ and $N_p/N = 53/105 = 50.5\%$; if $(p_2/p_1)_4 = -1$ (i.e., $p_2 \equiv 9 \bmod 16$), then $N_p = 52$ and $N_p/N = 52/105 = 49.5\%$.

D: For $p_1 \equiv p_2 \equiv 1 \bmod 4$ and $d_k < A_1 = 10{,}000$, and $p_1 = 2$ and $p_2 \equiv 1 \bmod 4$ (wlog) and $d_k < A_2 = 25{,}000$, let N_i correspond to A_i, i = 1, 2, and $N = N_1 + N_2$. Then $N = 452 + 215 = 667$ and if $N(e) = -1$ then $N_p = 292 + 145 = 437$ and $N_p/N = 437/667 \approx 65.5\%$.

Result 2: A: For $p_1 \equiv p_2 \equiv 1 \bmod 4$ and $d_k < A = 10{,}000$, $(p_1/p_2) = 1$, and $(p_1/p_2)_4 = (p_2/p_1)_4 = 1$: $N = 36$; if $N(e) = -1$ then $N_p = 13$ and $N_p/N = 13/36 \approx 36.1\%$.

B: For $p_1 = 2$ and $p_2 \equiv 1 \bmod 8$ and $d_k < A = 25{,}000$, and $(p_1/p_2)_4 = (p_2/p_1)_4 = 1$: $N = 24$; if $N(e) = -1$ then $N_p = 7$ and $N_p/N = 7/24 \approx 29.1\%$.

Remark 3: If we combine Results 2A and 2B then we obtain $N = 60$, $N_p = 20$, and $N_p/N = 20/60 = 1/3$, which agrees exactly with the prediction from Lemma 1.

In Table 1 below, for the reader's information we list the 20 fields we have found that comprise N_p in Results 2A and 2B, where $k = Q(\sqrt{}(p_1.p_2)$ as above.

Table 1. Fields k such that $(p_1/p_2) = (p_1/p_2)_4 = (p_2/p_1)_4 = 1$, $d_k < A = 10{,}000$ if $p_1 \equiv p_2 \equiv 1 \bmod 4$, $d_k < A = 25{,}000$ if $p_1 = 2$ and $p_2 \equiv 1 \bmod 8$, and $N(e) = -1$

$d_k \equiv 0 \bmod 8$	$d_k \equiv 1 \bmod 4$
8.113 = 904	5.461 = 2305
8.1201 = 9608	5.521 = 2605
8.1601 = 12808	5.541 = 2705
8.1777 = 14216	5.1061 = 5305
8.2113 = 16904	61.109 = 6649
8.3089 = 24712	29.233 = 6757
8.3121 = 24968	17.409 = 6953
	5.1601 = 8005
	41.197 = 8077
	17.509 = 8653
	5.1861 = 9305
	41.241 = 9881
	37.269 = 9953

4 Analysis of Heuristic Data in Support of Lemma 1

Our heuristic data in support of Lemma 1 is certainly quite minimal, and our work is just a preliminary heuristic investigation. However, from performing a chi-square statistical analysis [12,15] it is easily seen that our preliminary heuristic investigation supports Lemma 1 as well as Assumptions 1 and 2. The chi-square value used to determine if it is warranted to reject the null hypotheses described in Assumption 1, Assumption 2, and Lemma 1 was obtained from the following formula: x^2 = the summation taken from 1 to k of $(f_0 - f_e)^2/f_e$ where f_0 is the observed number in a given category, f_e is the expected number in that category, and the summation directs us to sum the ratio over all k categories [12,15]. To test the statements in Assumption 1, Assumption 2, and Lemma 1, we have $k = 2$ and a chi-square value of 3.84 is required to reject the null hypothesis at the common .05 level of significance [12,15]. For example, to test the statement in Assumption 1 that $P((p_1/p_2) = 1) = P((p_1/p_2) = -1) = .5$, we have $x^2 = (214 - 226)^2/226 + (238 - 226)^2/226 \approx 1.27 < 3.84$ and we therefore accept the null hypothesis. To test the statement in Assumption 2 that $P(N(e) = -1)) = 2/3$ we have $x^2 = (445 - 437)^2/437 + (222 - 230)^2/230 \approx .425 < 3.84$ and we therefore again accept the null hypothesis. To test the statement in Lemma 1 as described in Result 2A, we have $x^2 = (13-12)^2/12 + (23 - 24)^2/24 = .125 < 3.84$ and we therefore once again accept the null hypothesis. We leave it to the reader to obtain additional heuristic data to support Assumption 1 (see Remark 1) and to verify that applying the chi-square test to all of our theoretical hypotheses in Assumption 1, Assumption 2, and Lemma 1 results in the chi-square value not being sufficient to reject the null hypotheses.

5 Conclusion

Based upon our preliminary heuristic data we see that our results support Stevenhagen's conjecture (Assumption 2), our theoretical assumptions (Assumption 1), and our result that reinforces Stevenhagen's conjecture [1] assuming that $(p_1/p_2) = (p_1/p_2)_4 = (p_2/p_1)_4 = 1$ (Lemma 1), which may give us a useful probabilistic determination of how frequently $N(e) = -1$ occurs in this case. However, our heuristic research is based upon a relatively small number of low value discriminants and therefore further heuristic research is needed, with the goal of eventually proving Stevenhagen's conjecture, as described in both Assumption 2 and Lemma 1.

Competing Interests

Author has declared that no competing interests exist.

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