Journal of Scientific Research & Reports

14(3): 1-9, 2017; Article no.JSRR.32932
ISSN: 2320-0227

Semi-Analytical Method for the Computation of Nonlinear Second Order Differential Equations

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Authors' contributions

This work was carried out in collaboration between all authors. Author JS worked on the methodology adopted and its implementation on test problems using software. Authors MPA and JAK analyzed the basic properties of the method and also managed the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JSRR/2017/32932
Editor(s):
(1) Luigi Rodino, Professor of Mathematical Analysis, Dipartimento di Matematica, Università di Torino, Italy.
Reviewers:
(1) Shreekant Pathak, S. V. National Institute of Technology, India.
(2) Hytham A. Alkresheh, School of Mathematics, University of Science, Malaysia.
Complete Peer review History: http://www.sciencedomain.org/review-history/19053

Original Research Article

Received 23rd March 2017
Accepted 19th April 2017
Published 13th May 2017

ABSTRACT

This research paper considers a semi-analytical method called the Adomian Decomposition Method (ADM) for the computation of nonlinear second order differential equations. The analysis of the method will also be carried out to show the convergence of its solution. The method has the advantage of providing the solution to problems in a rapidly convergent series with components that are easily computable. The method also has an advantage of being continuous with no resort to discretization as is the case with most conventional methods. Numerical and graphical results shall also be presented to buttress out points.

Keywords: ADM; analysis; semi-analytical; series; nonlinear.

2010 Subject Classification: 65L05, 65L06, 65D30.

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1. INTRODUCTION

The semi-analytical method (i.e. the ADM) that will be considered in this paper provides solutions in a rapidly convergent series with easily computable components. One of the advantages of the method is that it can be used directly to solve all types of differential equations with homogeneous and inhomogeneous initial conditions. It also reduces the computational work in a tangible manner while maintaining higher accuracy of the numerical solution [1].

In this paper, the ADM shall be implemented on second order nonlinear problems of the form,

$$y''(t) = f(t,y),\ y(0) = y_0,\ y'(0) = y_1 \quad (1)$$

Where, y_0 *and* y_1 are real constants and $f(t,y)$ is a real-valued function. Suffice to say that problems of the form (1) find applications in many areas of human endeavor.

According to [2], most of the existing methods for solving problems of the form (1) are based on the principle of discretization (that is, they permit the generation of solutions only at selected points within the interval of integration). This deficiency leads to a situation where some fundamental properties of the problem to be solved are ignored. The ADM however, is a method that is continuous and permits integration at all interior points within the interval of integration.

A lot of work have been carried out on the solutions of differential equations using the ADM, see [2-7]. The authors in [2-5] studied the convergence analysis and implementation of ADM on some modeled differential equations while the authors in [4] worked on the optimal homotopy analysis methods for solving the linear and nonlinear Fokker-Planck equations. The author in [6] solved some frontier problems of physics using the ADM.

2. DERIVATION OF THE SEMI-ANALYTICAL METHOD

The ADM is a semi-analytical method whose crucial aspect is the employment of the ‘Adomian polynomials’ which allow for solution convergence of the nonlinear portion of the equation, without simply linearizing the system. These polynomials mathematically generalize to a Maclaurin series about an arbitrary external parameter; which gives the solution method more flexibility than direct Taylor series expansion, [8]. The method consists of splitting the given equation into linear and nonlinear parts, decomposing the unknown function into a series whose components are to be determined and also decomposing the nonlinear function in terms of the series solution by recurrent relation using the Adomian polynomials, [7].

Equation (1) can be written in an operator form as,

$$Ly = f(t,y) \quad (2)$$

The L in equation (2) is the differential operator defined by,

$$L = \frac{d^2}{dt^2} \quad (3)$$

In this context, the inverse operator L^{-1} is considered as a two-fold integral operator given by,

$$L^{-1} = \int_0^t \int_0^t (\bullet)\, dt dt \quad (4)$$

On the application of L^{-1} on both sides of equation (2) and imposing the initial conditions yield,

$$y(t) = y_0 + y_1(t) + L^{-1}\left[f(t,y) \right] \quad (5)$$

The ADM takes the form of a series solution for $y(t)$ given by an infinite sum of components,

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \quad (6)$$

Where, the components $y_n(t)$ is determined recursively. The method also define the nonlinear function $f(t,y)$ by the infinite series of polynomials given by,

$$f(t,y) = \sum_{n=0}^{\infty} A_n \quad (7)$$

On substituting (6) and (7) in (5), we get,

$$\sum_{n=0}^{\infty} y_n(t) = y_0 + y_1(t) + L^{-1}\left[\sum_{n=0}^{\infty} A_n\right] \tag{8}$$

The components of $y_n(t), n \geq 0$ are determined by firstly identifying the zeroth component $y_0(t)$ by all that arise from the initial conditions. The remaining components of the series (6) can be determined by using the preceding components.

Thus, each term of the series (6) is given by the recurrent relation,

$$\left.\begin{aligned} y_0(t) &= y_0 + y_1(t) \\ y_{n+1}(t) &= L^{-1}\left[A_n\right],\ n \geq 0 \end{aligned}\right\} \tag{9}$$

It is important to note that all the terms of the series (6) cannot be determined, thus the solution will be approximated by a series,

$$\phi_N(t) = \sum_{n=0}^{N-1} y_n(t) \tag{10}$$

Therefore, the ADM gives approximate series solution to the problem (1).

3. ANALYSIS OF THE SEMI-ANALYTICAL METHOD

The analysis of the semi-analytical method (i.e. the ADM) shall be reviewed in this section with reference to the works of [3] and [9] shall be considered.

Let us consider the functional equation,

$$y - N(y) = f \tag{11}$$

Where, N is a nonlinear operator from a Hilbert space H into H, f is a given function in H and we determine $y \in H$ satisfying equation (11).

Adomian considers the solution y to equation (11) as the sum of the series,

$$y = \sum_{i=0}^{\infty} y_n \tag{12}$$

The nonlinear operator is given by the following decomposed form,

$$N(y) = \sum_{n=0}^{\infty} A_n \tag{13}$$

Where, A_n's are polynomials in $y_0, y_1, ..., y_n$ called the Adomian polynomials.

The method consist of the following scheme,

$$\left.\begin{aligned} y_0 &= f \\ y_{n+1} &= A_n(y_0, y_1, y_2, ..., y_n) \end{aligned}\right\} \tag{14}$$

The ADM polynomial A_n's are obtained by,

$$n!A_n = \frac{d^n}{d\lambda^n}\left[N\left(\sum \lambda^i y_i\right)\right]_{\lambda=0},\ n = 0,1,2,... \tag{15}$$

The Adomian technique is equivalent to determining the sequence,

$$S_n = y_1 + y_2 + ... + y_n \tag{16}$$

Using the iterative scheme,

$$S_0 = 0,\ S_{n+1} = N\left(y_0 + S_n\right) \tag{17}$$

associated with the functional equation

$$S = N\left(y_0 + S\right) \tag{18}$$

Theorem 1 [9]:

Let N be an operator from a Hilbert space H into H and y be the exact solution of (11). Then, $\sum_{i=0}^{\infty} y_i$ which is obtained by (14), converges to y when there exist

$0 \leq \alpha < 1, \left\|y_{k+1}\right\| \leq \alpha \left\|y_k\right\|, \forall k \in N \cup \{0\}$.

Proof

We have

$S_0 = 0$
$S_1 = y_1$
$S_2 = y_1 + y_2$
.
.
.
$S_n = y_1 + y_2 + ... + y_n$

and we show that $\{S_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H. For this reason, consider

$$\|S_{n+1} - S_n\| = \|y_{n+1}\| \le \alpha \|y_n\| \le \alpha^2 \|y_{n-1}\| \le ... \le \alpha^{n+1} \|y_0\|$$

But for every $n, m \in N,\ n \ge m$, we have

$$\|S_n - S_m\| = \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{m+1} - S_m)\|$$
$$\le \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + ... + \|(S_{m+1} - S_m)\| \le \alpha^n \|y_0\| + \alpha^{n-1} \|y_0\| + ... + \alpha^{m+1} \|y_0\|$$
$$\le \left(\alpha^{m+1} + \alpha^{m+2} + ...\right) \|y_0\| = \frac{\alpha^{m+1}}{1-\alpha} \|y_0\|$$

Hence,

$\lim_{n,m \to +\infty} \|S_n - S_m\| = 0,$ that is $\{S_n\}_{n=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H and it implies that there exist $S \in H, \lim_{n \to +\infty} S_n = S$. That is, $S = \sum_{n=0}^{\infty} y_n$. But to solve equation (11) is equivalent to solving equation (18) and it implies that if N be a continuous operator, then

$$N(y_0 + S) = N\left(\lim_{n=+\infty} (y_0 + S_n)\right) = \lim_{n=+\infty} N(y_0 + S_n) = \lim_{n=+\infty} S_{n+1} = S$$

That is S is a solution of the equation (11) too. □

4. ORDER OF CONVERGENCE OF THE SEMI-ANALYTICAL METHOD

Definition 1: [10]

Let $\{S_n\}$ converge to S. If there exist two real positive constants p *and* C such that

$$\lim_{n \to \infty} \left| \frac{S_{n+1} - S}{(S_n - S)^p} \right| = C \tag{19}$$

then p is the order of convergence of $\{S_n\}$.

For determining the order of convergence of the sequence $\{S_n\}$, we consider the Taylor expansion of $N(y_0 + S_n)$.

$$N(y_0 + S_n) = N(y_0 + S) + \frac{N'(y_0 + S)}{1!}(S_n - S) + \frac{N''(y_0 + S)}{2!}(S_n - S)^2 + ... + \frac{N^{(k)}(y_0 + S)}{k!}(S_n - S)^k + ... \tag{20}$$

Using (17) and (18), we have

$$S_{n+1}-S=N'(y_0+S)(S_n-S)+\frac{N''(y_0+S)}{2!}(S_n-S)^2+...+\frac{N^{(k)}(y_0+S)}{k!}(S_n-S)^k+... \quad (21)$$

and state the following theorem.

Theorem 2 [10]**:**

Suppose $N\in C^p\left[a,b\right]$, if $N^{(k)}\left(y_0+S\right)=0,$ for $k=1,2,3,...,p-1$ and $N^{(p)}(y_0+S)\neq 0$, then the sequence $\left\{S_n\right\}$ is of order p.

Proof

By the hypothesis of the theorem, from (21) we have

$$S_{n+1}-S=\frac{N^{(p)}(y_0+S)}{p!}(S_n-S)^p+\frac{N^{(p+1)}(y_0+S)}{(p+1)!}(S_n-S)^{p+1}+...$$

Thus,

$$\frac{S_{n+1}-S}{(S_{n+1}-S)^p}=\frac{N^{(p)}(y_0+S)}{p!}+\frac{N^{(p+1)}(y_0+S)}{(p+1)!}(S_n-S)+...$$

Taking the limit when $n\longrightarrow\infty,$ completes the proof.

5. TEST PROBLEMS

The semi-analytical method will be implemented on two nonlinear problems of the form (1) with the aid of MATLAB 2015a programming language. The graphical plots shall also be generated using MATLAB 2015a.

Problem 5.1:

Consider the nonlinear problem,

$$y''(t)=8y^2(1+2t)^{-1},\ y(0)=1,\ y'(0)=-2 \quad (22)$$

The exact solution to (22) is given by,

$$y(t)=(1+2t)^{-1} \quad (23)$$

In operator form, we can rewrite the problem in () as,

$$Ly=8y^2(1+2t)^{-1} \quad (24)$$

Applying L^{-1} defined in (4) on both sides of equation (24) and imposing the initial conditions gives,

$$y(t)=1+(-2t)+L^{-1}\left[8y^2(1+2t)^{-1}\right] \quad (25)$$

We substitute the decomposition series (6) for $y(t)$ and the series of polynomials (7) for the nonlinear term y^2 into (25). This gives,

$$\sum_{n=0}^{\infty}y_n(t)=1+(-2t)+L^{-1}\left[8(1+2t)^{-1}\sum_{n=0}^{\infty}A_n\right] \quad (26)$$

The polynomials A_n in (26) are given by,

$$\begin{aligned}y^2&=(y_0+y_1+y_2+y_3+...)(y_0+y_1+y_2+y_3+...)\\&=y_0^2+2y_0y_1+2y_0y_2+y_1^2+2y_0y_3+2y_1y_2+...\end{aligned}$$

The following Adomian polynomials are obtained after collecting and rearranging terms,

$$\left.\begin{aligned} A_0 &= y_0^2 \\ A_1 &= 2y_0y_1 \\ A_2 &= 2y_0y_2 + y_1^2 \\ A_3 &= 2y_0y_3 + 2y_1y_2 \end{aligned}\right\} \tag{27}$$

The ADM gives the recursive relation,

$$\left.\begin{aligned} y_0(t) &= 1+(-2t) \\ y_{n+1}(t) &= L^{-1}\left[8(1+2t)^{-1}A_n\right], n \geq 0 \end{aligned}\right\} \tag{28}$$

Thus, the few components of $y_n(t)$ are given by,

$$\left.\begin{aligned} y_0(t) &= 1+(-2t) \\ y_1(t) &= -16t - 12t^2 + \frac{8}{3}t^3 + 8\log(1+2t) + 16t\log(1+2t) \\ y_2(t) &= -128t - \frac{640}{3}t^2 + \frac{128}{3}t^3 + \frac{176}{9}t^4 - \frac{32}{15}t^5 + 64\log(1+2t) - \frac{64}{3}t(-8-3t+2t^2)\log(1+2t) \\ &. \\ &. \\ &. \end{aligned}\right\} \tag{29}$$

Substituting the terms in equation (29) into equation (10), we obtain the series solution,

$$\begin{aligned} \phi_N(t) = \sum_{n=0}^{N-1} y_n(t) &= 1 - 2t - 16t - 12t^2 + \frac{8}{3}t^3 + 8\log(1+2t) + 16t\log(1+2t) - 128t - \frac{640}{3}t^2 \\ &+ \frac{128}{3}t^3 + \frac{176}{9}t^4 - \frac{32}{15}t^5 + 64\log(1+2t) - \frac{64}{3}t(-8-3t+2t^2)\log(1+2t) + \ldots \end{aligned} \tag{30}$$

Using the first ten terms in (30), we obtain the approximate solution for the problem in equation (10) as presented in Table 1.

Problem 5.2:

Consider the nonlinear problem,

$$y''(t) = -e^{-2y},\ y(0) = 1,\ y'(0) = \frac{1}{e} \tag{31}$$

The exact solution to (27) is given by,

$$y(t) = \ln(t+e) \tag{32}$$

In operator form, we can rewrite the problem in (31) as,

$$Ly = -e^{-2y} \tag{33}$$

Applying L^{-1} defined in (4) on both sides of equation (33) and imposing the initial conditions gives,

$$y(t) = 1 + \frac{t}{e} - L^{-1}\left[e^{-2y}\right] \tag{34}$$

We substitute the decomposition series (6) in equation (34). This gives,

$$\sum_{n=0}^{\infty} y_n(t) = 1 + \frac{t}{e} + L^{-1}\left[e^{-2y}\right] \quad (35)$$

This gives the following Adomian polynomials,

$$\left.\begin{aligned} A_0 &= -e^{-2y_0(t)} \\ A_1 &= 2y_1(t)e^{-2y_0(t)} \\ A_2 &= -2e^{-2y_0(t)}y_1(t)^2 + 2y_2(t)e^{-2y_0(t)} \end{aligned}\right\} \quad (36)$$

The ADM gives the following recursive relation,

$$\left.\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= \frac{t}{e} + L^{-1}(A_0) \\ y_{n+1}(t) &= L^{-1}\left[A_n\right], n \geq 1 \end{aligned}\right\} \quad (37)$$

The slight modification is to avoid some difficult terms in the Adomian polynomial A_n that contains $e^{-2y_0(t)}$.

Thus, the few components of $y_n(t)$ are given by,

$$\left.\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= \frac{t}{e} - \frac{t^2}{2e^2} \\ y_2(t) &= \frac{\left(\frac{t^3}{3e} - \frac{t^4}{12e^2}\right)}{e^2} \\ y_3(t) &= -\frac{t^4}{6e^4} + \frac{2t^5}{15e^5} - \frac{t^6}{45e^6} \\ &\vdots \end{aligned}\right\} \quad (38)$$

Substituting the terms in equation (38) into equation (10), we obtain the series solution,

$$\phi_N(t) = \sum_{n=0}^{N-1} y_n(t) = 1 + \frac{t}{e} - \frac{t^2}{2e^2} + \frac{\left(\frac{t^3}{3e} - \frac{t^4}{12e^2}\right)}{e^2} - \frac{t^4}{6e^4} + \frac{2t^5}{15e^5} - \frac{t^6}{45e^6} + \ldots \quad (39)$$

Using the first ten terms in (39), we obtain the approximate solution for the problem in equation (10) as presented in Table 2.

Table 1. Showing the results for problem 5.1

t	**Exact solution**	**Computed solution**	**Error**	**Time/s**
0.1000	8.333333333333334e-001	8.333333333330000e-001	3.334e-012	0.0101
0.2000	7.142857142857143e-001	7.142857142857042e-001	1.010e-012	0.0109
0.3000	6.250000000000000e-001	6.250000000002567e-001	2.567e-012	0.0112
0.4000	5.555555555555556e-001	5.555555555554956e-001	6.000e-012	0.0117
0.5000	5.000000000000000e-001	5.000000000002417e-001	2.417e-012	0.0121
0.6000	4.545454545454545e-001	4.545454545454870e-001	3.250e-012	0.0128
0.7000	4.166666666666667e-001	4.166666666662345e-001	4.322e-012	0.0130
0.8000	3.846153846153846e-001	3.846153846153745e-001	1.010e-012	0.0133
0.9000	3.571428571428572e-001	3.571428571428242e-001	7.000e-012	0.0138
1.0000	3.333333333333333e-001	3.333333333332956e-001	3.770e-012	0.0142

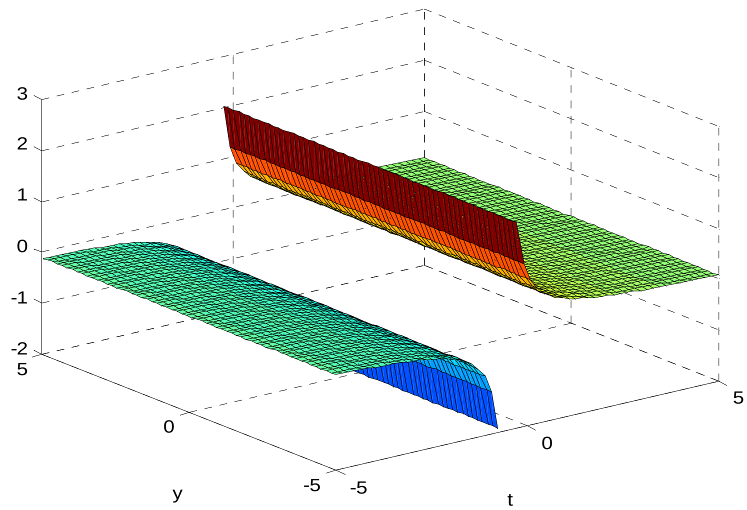

Fig. 1. Graphical result showing the nonlinear nature of Problem 5.1

Table 2. Showing the results for problem 5.2

t	Exact solution	Computed solution	Error	Time/s
0.1000	1.036127418574834e+000	1.036127418574952e+000	1.180e-012	0.0278
0.2000	1.070995028184911e+000	1.070995028184859e+000	5.200e-012	0.0282
0.3000	1.104687738510315e+000	1.104687738510458e+000	1.430e-012	0.0287
0.4000	1.137282154125937e+000	1.137282154126000e+000	6.300e-012	0.0290
0.5000	1.168847623498306e+000	1.168847623498212e+000	9.400e-012	0.0293
0.6000	1.199447127419420e+000	1.199447127419399e+000	2.100e-012	0.0297
0.7000	1.229138035495247e+000	1.229138035495423e+000	1.760e-012	0.0301
0.8000	1.257972753527120e+000	1.257972753527220e+000	1.000e-012	0.0309
0.9000	1.285999280141410e+000	1.285999280141500e+000	9.000e-012	0.0311
1.0000	1.313261687518223e+000	1.313261687518326e+000	1.030e-012	0.03.18

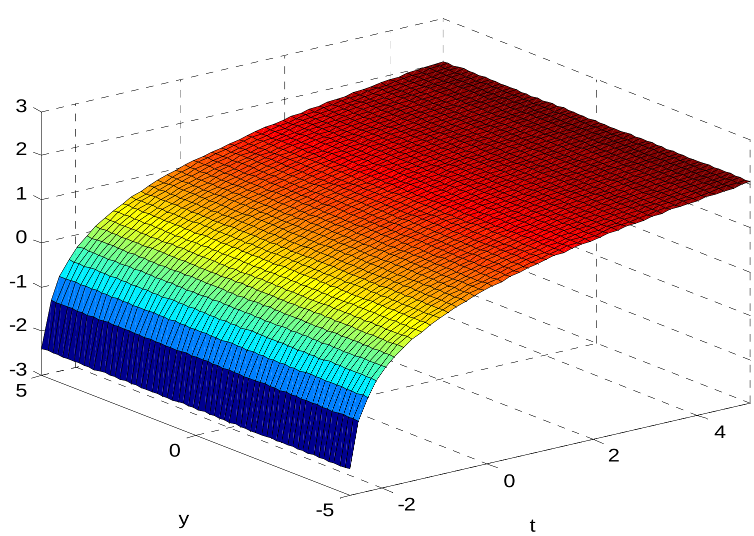

Fig. 2. Graphical result showing the nonlinear nature of Problem 5.2

6. CONCLUSION

The semi-analytical method derived was implemented on two nonlinear problems. The method, called the ADM, was also analyzed with the view to studying its convergence properties. The results obtained buttress the fact that the method is convergent. The evaluation time per seconds is also very small (see Tables 1 and 2) implying that the method is efficient and computationally reliable.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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    DOI: 10.1016/s0096-3003(01)00103-5.

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/19053