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The Length of the Generalized Bloch Vector

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

In this paper we calculate the Bloch vector length with respect to the eigenvalues of the density matrix and the mixed states.

Keywords: Bloch vector.

1 INTRODUCTION

For a qubit (two level quantum system) a density matrix can be expressed by a 3-dimensional vector, the Bloch vector, and any such vector has to lie within the so-called Bloch ball. The inside the ball corresponds to a physical state, i.e. a density matrix. The pure states lie on the sphere and the mixed ones inside.

In this article we consider n-level quantum systems and present the length of the generalized Bloch vector with respect to the eigenvalues of the density matrix.

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2 SOME RESULTS

Any density matrix ρ of n-level system can be expressed as a linear combination of matrices λ_i called generalized Pauli matrices :

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\mathbf{r}\cdot\lambda\right),$$

where $\mathbf{r} = (x_1, x_2, \cdots, x_{n^2-1}) \in \mathbb{R}^{n^2-1}$ is the generalized Bloch vector. To find the eigenvalues τ of the density matrix ρ, we should solve $\det(\tau I - \rho) = 0$, where $\det(\tau I - \rho)$ is the characteristic polynomial of ρ. In the general case the characteristic polynomial is

$$\begin{aligned}\det(\tau I - \rho) = \tau^n + c_1\tau^{n-1} + c_2\tau^{n-2} \\ + \cdots + c_{n-1}\tau + c_n\end{aligned} \tag{2.1}$$

and we consider the coefficient c_2 at τ^{n-2}. c_2 is given by [1] and [2] :

$$\begin{aligned}c_2 &= \frac{1}{2}\left((\text{Tr }\rho)^2 - \text{Tr}(\rho^2)\right) \\ &= \frac{1}{2}\left(1 - \text{Tr}(\rho^2)\right),\end{aligned}$$

since for density matrices Tr $\rho = 1$. According to [1] we have

$$\text{Tr}(\rho^2) = \frac{1}{n}\left(1 + (n-1)|\mathbf{r}|^2\right)$$

and

$$c_2 = \frac{n-1}{2n}\left(1 - |\mathbf{r}|^2\right).$$

Let $\{\tau_1, \tau_2, \cdots, \tau_n\}$ be the roots of (2.1) and thus the eigenvalues of ρ. We note that

$$c_1 = \sum_{i=1}^{n}\tau_i = 1 \tag{2.2}$$

and we express c_2 as

$$\begin{aligned}c_2 &= \sum_{1\le i<j\le n}\tau_i\tau_j \\ &:= P_2(\tau_1, \tau_2, \cdots, \tau_n),\end{aligned} \tag{2.3}$$

where P_2 is the second order symmetric polynomial of n variables. From the equation (2.3) we obtain the following results :

Lemma 2.1. *Let $P_2(\tau_1, \tau_2, \cdots, \tau_n)$ be as in (2.3) and m be a real number. Then*

(a)

$$\begin{aligned}&P_2(m\tau_1, m\tau_2, \cdots, m\tau_n) \\ &= m^2 P_2(\tau_1, \tau_2, \cdots, \tau_n),\end{aligned}$$

(b)

$$\begin{aligned}&P_2(\tau_1 + m, \tau_2 + m, \cdots, \tau_n + m) \\ &= P_2(\tau_1, \tau_2, \cdots, \tau_n) \\ &\quad + \frac{m(n-1)(mn+2)}{2},\end{aligned}$$

(c)

$$\begin{aligned}&P_2(\tau_1^m, \tau_2^m, \cdots, \tau_n^m) \\ &= \frac{1}{2}\left\{\left(\sum_{i=1}^{n}\tau_i^m\right)^2 - \sum_{i=1}^{n}\tau_i^{2m}\right\}.\end{aligned}$$

Proof. (a) By (2.3) we easily observe that

$$\begin{aligned}&P_2(m\tau_1, m\tau_2, \cdots, m\tau_n) \\ &= \sum_{1\le i<j\le n} m\tau_i \cdot m\tau_j \\ &= m^2\sum_{1\le i<j\le n}\tau_i\tau_j \\ &= m^2 P_2(\tau_1, \tau_2, \cdots, \tau_n).\end{aligned}$$

(b) From (2.3) we have

$$\begin{aligned}&P_2(\tau_1 + m, \tau_2 + m, \cdots, \tau_n + m) \\ &= \sum_{1\le i<j\le n}(\tau_i + m)(\tau_j + m) \\ &= \sum_{1\le i<j\le n}\left(\tau_i\tau_j + m\tau_i + m\tau_j + m^2\right) \\ &= \sum_{1\le i<j\le n}\tau_i\tau_j + m\sum_{1\le i<j\le n}\tau_i \\ &\quad + m\sum_{1\le i<j\le n}\tau_j + m^2\sum_{1\le i<j\le n}1.\end{aligned} \tag{2.4}$$

Then by (2.2) we obtain

$$
\begin{aligned}
& m \sum_{1 \le i < j \le n} \tau_i + m \sum_{1 \le i < j \le n} \tau_j \\
&= m\{\tau_1(n-1) + \tau_2(n-2) + \cdots \\
&\quad + \tau_{n-1}\} \\
&\quad + m\{\tau_2 + 2\tau_3 + \cdots + (n-1)\tau_n\} \\
&= m(n-1)(\tau_1 + \tau_2 + \cdots + \tau_n) \\
&= m(n-1)
\end{aligned}
$$

and

$$
\begin{aligned}
& m^2 \sum_{1 \le i < j \le n} 1 \\
&= m^2\{(n-1) + (n-2) + \cdots + 1\} \\
&= m^2 \cdot \frac{(n-1)n}{2}.
\end{aligned}
$$

Applying the above results into Eq. (2.4) we deduce that

$$
\begin{aligned}
& P_2(\tau_1 + m, \tau_2 + m, \cdots, \tau_n + m) \\
&= \sum_{1 \le i < j \le n} \tau_i \tau_j + m(n-1) \\
&\quad + m^2 \cdot \frac{(n-1)n}{2} \\
&= P_2(\tau_1, \tau_2, \cdots, \tau_n) \\
&\quad + \frac{m(n-1)(mn+2)}{2}.
\end{aligned}
$$

(c) Since

$$
\begin{aligned}
& \left(\sum_{i=1}^{n} \tau_i^m \right)^2 \\
&= (\tau_1^m + \tau_2^m + \cdots + \tau_n^m)^2 \\
&= \tau_1^{2m} + \tau_2^{2m} + \cdots + \tau_n^{2m} \\
&\quad + 2P_2(\tau_1^m, \tau_2^m, \cdots, \tau_n^m) \\
&= \sum_{i=1}^{n} \tau_i^{2m} + 2P_2(\tau_1^m, \tau_2^m, \cdots, \tau_n^m),
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
& P_2(\tau_1^m, \tau_2^m, \cdots, \tau_n^m) \\
&= \frac{1}{2} \left\{ \left(\sum_{i=1}^{n} \tau_i^m \right)^2 - \sum_{i=1}^{n} \tau_i^{2m} \right\}.
\end{aligned}
$$

□

Let $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$. Then the Bloch vector $\mathbf{r}$ of n-level system [3] satisfies

$$
|\mathbf{r}|^2 = \frac{n}{n-1} \left(|\tau|^2 - \frac{1}{n} \right). \qquad (2.5)
$$

Proposition 2.1. *The length of the Bloch vector $|\mathbf{r}|$ for state ρ is proportional to the distance between $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ and $\nu = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ in the eigenvalue simplex, where τ_i are the eigenvalues of ρ.*

Proof. See [3].

□

As an application of Proposition 2.1, we consider the following theorem :

Theorem 2.2. *The length of the Bloch vector $|\mathbf{r}|$ for state ρ is*

$$
|\mathbf{r}| = \sqrt{\frac{2n^3|\tau - \nu|^2 - (2n-1)(n-1)}{2n^2(n-1)}},
$$

where $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ and the mixed state $\nu = (\frac{1}{n^2}, \frac{1}{n^2}, \cdots, \frac{1}{n^2})$. Note that τ_i are the eigenvalues of ρ.

Proof. Let $\delta = \tau - \nu$ and the mixed state $\nu = (\frac{1}{n^2}, \frac{1}{n^2}, \cdots, \frac{1}{n^2})$. Then by (2.2) we have

$$
\begin{aligned}
\sum_{i=1}^{n} \delta_i &= \sum_{i=1}^{n} \tau_i - \sum_{i=1}^{n} \nu_i \\
&= 1 - \sum_{i=1}^{n} \frac{1}{n^2} \\
&= 1 - \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \\
&= \frac{1}{2} - \frac{1}{2n}.
\end{aligned}
$$

According to (2.5) and the above identity we deduce that

$$|\mathbf{r}|^2 = \frac{n}{n-1}\left(|\tau|^2 - \frac{1}{n}\right)$$
$$= \frac{n}{n-1}\left(|\delta+\nu|^2 - \frac{1}{n}\right)$$
$$= \frac{n}{n-1}\left(\sum_{i=1}^{n}(\delta_i+\nu_i)^2 - \frac{1}{n}\right)$$
$$= \frac{n}{n-1}\left(\sum_{i=1}^{n}(\delta_i+\frac{1}{n^2})^2 - \frac{1}{n}\right)$$
$$= \frac{n}{n-1}\left(\sum_{i=1}^{n}\delta_i^2 + 2\frac{1}{n^2}\sum_{i=1}^{n}\delta_i + \frac{1}{n^4}\sum_{i=1}^{n}1 - \frac{1}{n}\right)$$

and so

$$|\mathbf{r}|^2 = \frac{n}{n-1}\left\{\sum_{i=1}^{n}\delta_i^2 + 2\frac{1}{n^2}\left(\frac{1}{2}-\frac{1}{2n}\right) + \frac{1}{n^4}\cdot\frac{n(n+1)}{2} - \frac{1}{n}\right\}$$
$$= \frac{n}{n-1}\left(|\delta|^2 + \frac{3}{2n^2} - \frac{1}{2n^3} - \frac{1}{n}\right).$$

Finally we have

$$|\mathbf{r}| = \sqrt{\frac{n}{n-1}\left(|\delta|^2 + \frac{3}{2n^2} - \frac{1}{2n^3} - \frac{1}{n}\right)}.$$

□

It is known that there are no valid states such that $|\mathbf{r}| > 1$. Pure states have $|\mathbf{r}| = 1$, but mixed states have $|\mathbf{r}| < 1$. Thus to fit the mixed states we range $|\tau-\nu|$ from $0.5 \sim 0.81$. As shown in Fig. 1, 2, and 3, the Bloch vector hardly affect by n even though n extends to a large number.

However when we alter $|\tau-\nu|$ a little, the Bloch vector changes rapidly in Fig. 4 and 5 comparing with the above pictures.

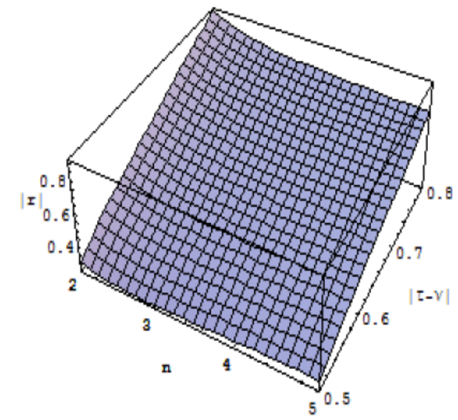

Fig. 1. $|\mathbf{r}|$ **versus** n **(**$2 \le n \le 5$**) and** $|\tau-\nu|$ **(**$0.5 \le |\tau-\nu| \le 0.8$**)**

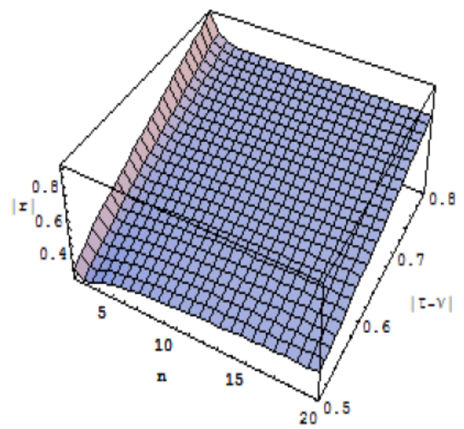

Fig. 2. $|\mathbf{r}|$ **versus** n **(**$2 \le n \le 20$**) and** $|\tau-\nu|$ **(**$0.5 \le |\tau-\nu| \le 0.8$**)**

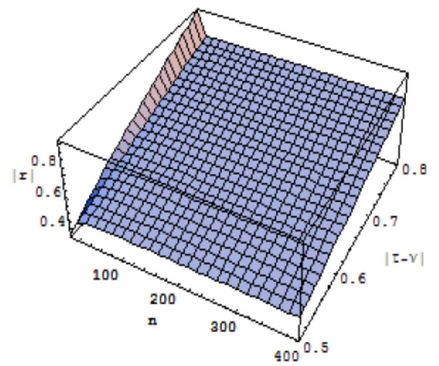

Fig. 3. $|\mathbf{r}|$ **versus** n **(**$2 \le n \le 400$**) and** $|\tau-\nu|$ **(**$0.5 \le |\tau-\nu| \le 0.8$**)**

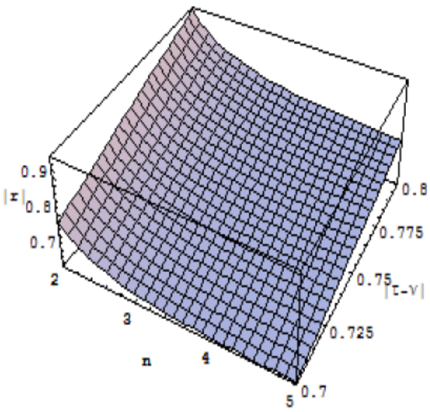

Fig. 4. $|\mathbf{r}|$ **versus** n **(**$2 \le n \le 5$**) and** $|\tau - \nu|$ **(**$0.7 \le |\tau - \nu| \le 0.8$**)**

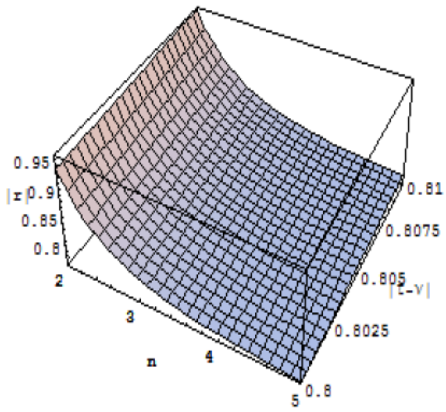

Fig. 5. $|\mathbf{r}|$ **versus** n **(**$2 \le n \le 5$**) and** $|\tau - \nu|$ **(**$0.8 \le |\tau - \nu| \le 0.81$**)**

Remark 2.1. From Theorem 2.2 we can obtain

$$\begin{aligned}&\lim_{n\to\infty} |\mathbf{r}| \\ &= \lim_{n\to\infty} \sqrt{\frac{2n^3|\tau-\nu|^2-(2n-1)(n-1)}{2n^2(n-1)}} \\ &= |\tau-\nu|,\end{aligned}$$

which implies that for the mixed state the Bloch vector is proportional to the distance between $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ and $\nu = (\frac{1}{n^2}, \frac{1}{n^2}, \cdots, \frac{1}{n^2})$.

Theorem 2.3. *Let* $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ *satisfying* $\tau_1 = \tau_n$, $\tau_2 = \tau_{n-1}$, $\tau_3 = \tau_{n-2}$, $\cdots$, *that is,* $\tau_i = \tau_{n-i+1}$ *for* $1 \le i \le n$ *and* $\nu = (\frac{1}{n^2}, \frac{2}{n^2}, \cdots, \frac{n}{n^2})$. *Then the length of the Bloch vector* $|\mathbf{r}|$ *for state* ρ *is*

$$|\mathbf{r}| = \sqrt{\frac{6n^3|\tau-\nu|^2-(2n-1)(n-1)}{6n^2(n-1)}}.$$

Note that τ_i *are the eigenvalues of* ρ.

Proof. Let $\delta = \tau - \nu$ and the mixed state $\nu = (\frac{1}{n^2}, \frac{2}{n^2}, \cdots, \frac{n}{n^2})$. Then we can know that

$$\begin{aligned}\sum_{i=1}^n i\delta_i &= \sum_{i=1}^n i\tau_i - \sum_{i=1}^n i\nu_i \\ &= \sum_{i=1}^n i\tau_i - \sum_{i=1}^n i \cdot \frac{i}{n^2} \\ &= \frac{n+1}{2} - \frac{1}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n}{6} - \frac{1}{6n}\end{aligned}$$

since by the assumption $\tau_i = \tau_{n-i+1}$ and (2.2) we deduce that

$$\begin{aligned}&\sum_{i=1}^n i\tau_i \\ &= \tau_1 + 2\tau_2 + \cdots + (n-1)\tau_{n-1} + n\tau_n \\ &= n\sum_{i=1}^n \tau_i - \{(n-1)\tau_1 + (n-2)\tau_2 + \\ &\quad \cdots + \tau_{n-1}\} \\ &= n\sum_{i=1}^n \tau_i - \{(n-1)\tau_n + (n-2)\tau_{n-1} \\ &\quad + \cdots + \tau_2\} \\ &= n\sum_{i=1}^n \tau_i - \sum_{i=1}^n (i-1)\tau_i \\ &= (n+1)\sum_{i=1}^n \tau_i - \sum_{i=1}^n i\tau_i \\ &= (n+1) - \sum_{i=1}^n i\tau_i\end{aligned}$$

and so

$$\sum_{i=1}^{n} i\tau_i = \frac{n+1}{2}.$$

According to (2.5) and the above results we have

$$\begin{aligned}|\mathbf{r}|^2 &= \frac{n}{n-1}\left(|\tau|^2 - \frac{1}{n}\right)\\ &= \frac{n}{n-1}\left(|\delta+\nu|^2 - \frac{1}{n}\right)\\ &= \frac{n}{n-1}\left(\sum_{i=1}^{n}(\delta_i+\nu_i)^2 - \frac{1}{n}\right)\\ &= \frac{n}{n-1}\left(\sum_{i=1}^{n}(\delta_i+\frac{i}{n^2})^2 - \frac{1}{n}\right)\\ &= \frac{n}{n-1}\left(\sum_{i=1}^{n}\delta_i^2 + 2\frac{1}{n^2}\sum_{i=1}^{n} i\delta_i\right.\\ &\qquad \left. + \frac{1}{n^4}\sum_{i=1}^{n} i^2 - \frac{1}{n}\right)\end{aligned}$$

and so

$$\begin{aligned}|\mathbf{r}|^2 &= \frac{n}{n-1}\left\{\sum_{i=1}^{n}\delta_i^2 + 2\frac{1}{n^2}\left(\frac{n}{6} - \frac{1}{6n}\right)\right.\\ &\qquad \left. + \frac{1}{n^4}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{n}\right\}\\ &= \frac{n}{n-1}\left(|\delta|^2 + \frac{1}{2n^2} - \frac{1}{6n^3} - \frac{1}{3n}\right).\end{aligned}$$

Thus we conclude that

$$|\mathbf{r}| = \sqrt{\frac{n}{n-1}\left(|\delta|^2 + \frac{1}{2n^2} - \frac{1}{6n^3} - \frac{1}{3n}\right)}.$$

□

3 CONCLUSION

We calculate the length of the Bloch vector for the special mixed state cases in Theorem 2.2, Remark 2.1, and Theorem 2.3.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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