British Journal of Mathematics & Computer Science
22(1): 1-18, 2017; Article no.BJMCS.33168
ISSN: 2231-0851

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Analytic Approximation Solutions of Lyapunov Orbits around the Collinear Equilibrium Points for Binary α-Centuari System: The Planar Case

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Authors' contributions

This work was carried out in collaboration between both authors. Authors JS and JMG designed the study, managed the analyses of the study, wrote the protocol, managed the literature searches and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2017/33168
Editor(s):
(1) Serkan Araci, Hasan Kalyoncu University, Turkey.
Reviewers:
(1) Jagdish Prakash, University of Botswana, Botswana.
(2) Abdullah Sonmezoglu, Bozok University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/18799

Received: 1st April 2017
Accepted: 12th April 2017
Published: 26th April 2017

Original Research Article

Abstract

A third order analytic approximation solution of Lyapunov orbits around the collinear equilibrium in the planar restricted three-body problem by utilizing the Lindstedt Poincaré method is presented. The primaries are oblate bodies and sources of radiation pressure. The theory has been applied to the binary α-Centuari system in six cases. Also, we have determined numerically the positions of the collinear equilibrium points and shown the effects of the parameters concerned with these equilibrium points.

Keywords: Approximate solutions; periodic orbit; RTBP.

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1 Introduction

The restricted three body problem (R3BP) is an instance whereby two bodies (known as primaries) which have significant masses as compared to a third body (known as the infinitesimal body) with negligible mass, move in circular orbits about their common barycentre. And the motion of the third body is influenced by the gravitational attraction of the primaries; whereas the motion of the primaries is not affected by the gravitational field of the third boy. As such, this problem can also be viewed as a special case of the two body problem (2BP).

Models for the R3BP can also be taken from the stellar systems [1,2]. Studies with applications to the binary star systems enable scientists to determine the mass of a star by the calculations of their orbits. This in turn allows other astronomical parameters like size, temperature, radius and density of the double stars to be determined by astronomers.

So far, the R3BP has been shown to have only particular solutions. One of such is the five stationary or equilibrium points (three of which lie on the line joining the primaries called the collinear equilibrium points and are denoted as L_1 , L_2 and L_3 while the other two which form triangular configurations with the primaries and known as the triangular equilibrium points are represented as L_4 and L_5). Another particular solution is the periodic orbits around the equilibrium points or around the primary bodies.

Apart from the classification of orbits in periodic solutions among several other uses, studies on periodic orbits are valuable when it comes to station keeping and launching of artificial satellite. Over the years, studies have been carried out on periodic orbits around the equilibrium points in the R3BP, in-plane or perpendicular to the plane of motion [3,4,5,6,7,8,9,10,11,12,13]. These investigations involved the use of either analytical, numerical or a combination of both methods. The analytical methods provide approximate solutions or exact solution to the problem. From these solutions, the infinitesimal or starting orbits near the equilibrium points are obtained. To continue to families of periodic orbits around the collinear equilibrium points or the triangular points or the primary bodies, researchers make use of the numerical application method known as the differential corrections scheme.

In order to examine some perturbing effects (radiation pressure, Poynting-Robertson drag, solar wind drag, Coriolis and centrifugal forces and angular velocity) and to consider the non-spherical nature (oblateness and triaxiallity) of the primaries, some researchers have made modifications to the classical R3BP in their studies of periodic orbits around the equilibrium points (specifically for this study, the collinear equilibrium points). Some of such works can be seen in [14,15,16,17,18,19,20,21,22,23,24,25].

Richardson [26] gave a third order analytical solution for halo-type periodic motion about the collinear points of the R3BP by utilizing the method of successive approximations in conjunction with a technique similar to the Lindstedt-Poincaré method with application to Sun-Earth system. Also, by giving an analytical approximation to periodic orbits in the circular restricted three body problem (CR3BP), Nagel-Pichardo [27] derived a simple set of analytical expressions that give periodic orbits on the disc of binary systems without the need to solve the equations of motion by numerical integration.

In their work on periodic solutions in the CR3BP, Gao-Zhang [28] presented an analytical expression of periodic solutions of the first-order approximate system. Pal-Kushvah [29] gave a third order analytic approximation solution of halo and Lissajous orbits when they considered the effect of radiation pressure, Poynting-Robertson drag and solar wind drag on the Sun-(Earth-Moon) R3BP.

In this study, we give a third order analytic approximation of periodic solution around the collinear equilibrium points in the planar circular restricted three body problem (PCR3BP) by utilizing the Lindstedt-Poincaré method. From the approximate periodic solution, the initial conditions or starting orbits near the collinear points have been obtained. The theory is applied to the binary α-Centuari system, where α-Centuari A is the primary and α-Centuari B is the secondary. The infinitesimal body is taken to be a possible exoplanet with negligible mass as compared to the masses of the primaries moving in the plane of motion of the binary system. The assumptions made here are that both stars have significant radiation effects and are sufficiently oblate in shape. Arbitrary chosen values for the radiation and oblateness coefficients have been drawn in six instances (cases). In the first case, the binary system is considered with respect to the classical problem (spherical nature of the primaries) while the other five cases have values for all the parameters concerned.

In the second section of this study, we give the equations of motion of the problem and show the regions of possible motion of the infinitesimal body. In the next section, we numerically determine the positions of the collinear equilibrium points and show the effects of the radiation pressure and oblateness parameters on these points. In sect. 4, we give the third order Lindstedt-Poincaré local analysis of Lyapunov orbits around the collinear equilibrium points, and present the numerical results in sect. 5, while we give the conclusions in the last section.

2 Equations of Motion

Let m_1, m_2 and m be the masses of the primary, secondary and infinitesimal bodies (that is, α-Centuari A, α-Centuari B and an exoplanet), respectively. Here, the primary bodies are moving in circular orbits about their common barycentre, while the infinitesimal body is moving and exerting no influence in the plane of motion of the primaries. The mass parameter is given by $\mu = \dfrac{m_2}{m_1 + m_2}$. Let the unit of distance be taken as the distance between the primaries, such that the gravitational constant $G = 1$. The unit of mass has been chosen so that $m_1 + m_2 = 1$ and we take $m_1 = 1 - \mu$ and $m_2 = \mu$. We let Oxy be the synodic coordinate system with the position of the infinitesimal body as $P(x, y)$ and the primary and secondary bodies as $P_1(\mu, 0)$ and $P_2(-(1-\mu), 0)$ respectively. Thus, the equations of motion of the infinitesimal body in the dimensionless synodic coordinate system with radiation pressure parameters q_1 and $q_2 (q_i \le 1, i = 1, 2)$ and oblateness parameters A_1 and $A_2 (A_i \ll 1, i = 1, 2)$ [30] are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \end{aligned} \tag{1}$$

with

$$\Omega = \frac{n^2}{2}(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1 q_1}{2r_1^3} + \frac{\mu A_2 q_2}{2r_2^3},$$

where

$$\begin{aligned} r_1 &= \sqrt{(x-\mu)^2 + y^2}, \\ r_2 &= \sqrt{(x-\mu+1)^2 + y^2}, \end{aligned}$$

and n is the mean motion, given as

$$n = \sqrt{1 + \frac{3}{2}(A_1 + A_2)} \,.$$

The Jacobi integral which is obtained from eq. (1) is given by

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C \,,$$

where the symbol C denotes the Jacobi Constant.

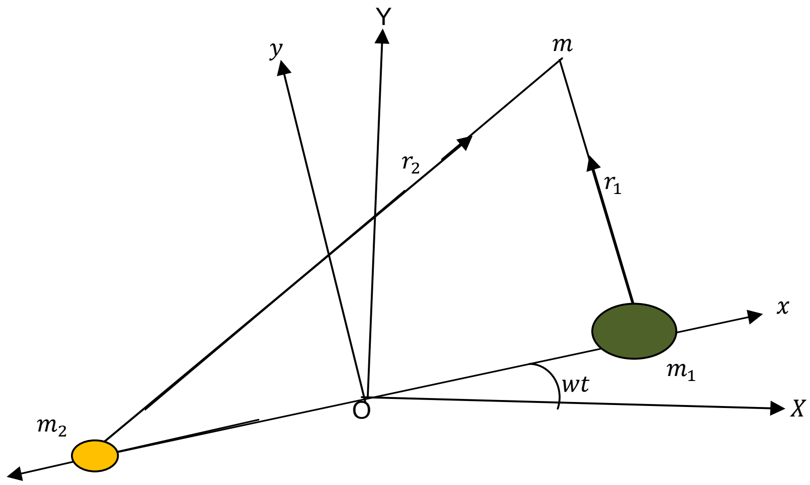


Fig. 1. The configuration of the rotating coordinate system for the restricted three-body problem where m_1, m_2 and m are the oblate primaries and infinitesimal body respectively

The Jacobian integral is used to obtain the Zero-velocity surface plots by assuming that the velocity variables are equal to zero. This surface divides the space into two regions. One of the regions is known as the region of possible motion while the other is called the forbidden region. These regions describe the area where the infinitesimal body is allowed and where it is not allowed. In Fig. 3 there are three distinct curves which represent the Zero-velocity curves for the Jacobi Constant when the first, second and third collinear equilibrium points are considered with respect to the present model. Within these curves are the forbidden regions of motion for the infinitesimal body.

The actual masses of the stars α-Centuari A and α-Centuari B are 2.192×10^{30} Kg and 1.970×10^{30} Kg respectively. Then, the mass parameter $\mu = \frac{1.970}{4.162} = 0.4733 < \frac{1}{2}$.

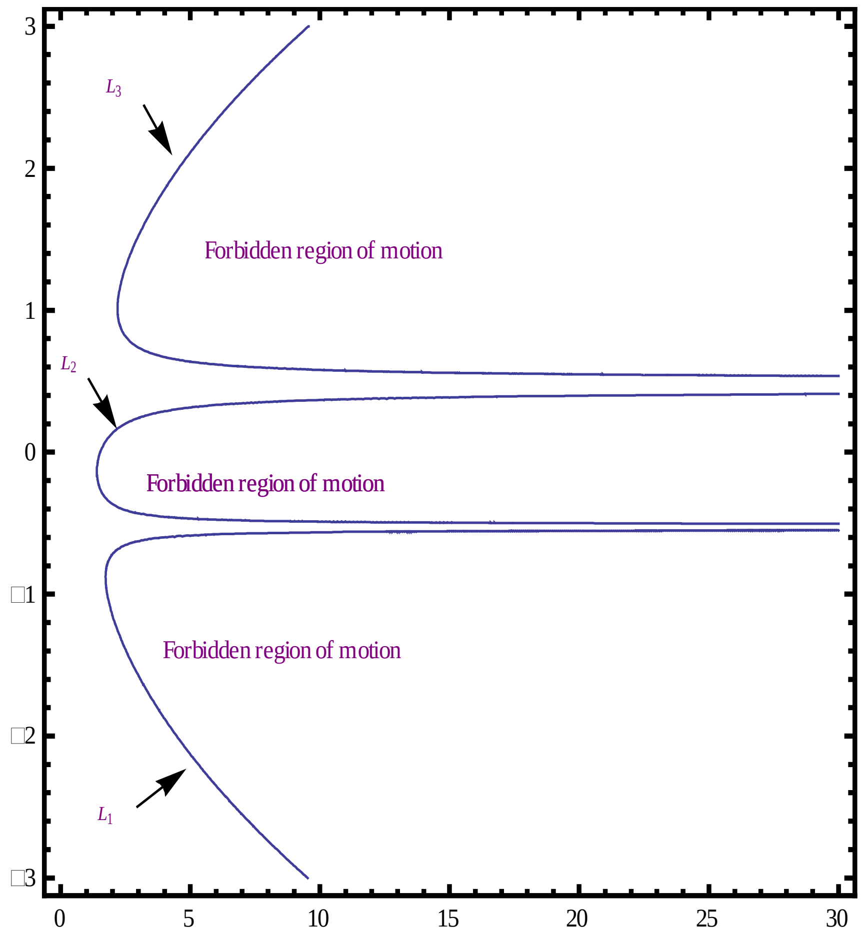


Fig. 2. The forbidden region of the infinitesimal body with respect to the Jacobian constant C and the collinear equilibrium points

3 Determination of the Collinear Equilibrium Points

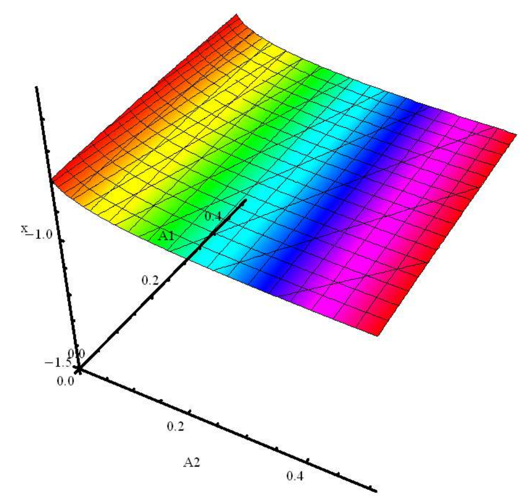
The positions of the collinear equilibrium points are obtained from the solution of the nonlinear algebraic equation $\Omega_x = 0$, when $y = 0$ by solving for x. That is, we solve

$$n^2 x - \frac{q_1(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu q_2(x+1-\mu)}{|x-\mu|^3} - \frac{3A_1(1-\mu)(x-\mu)q_1}{2|x-\mu|^5} - \frac{3\mu A_2(x+1-\mu)q_2}{2|x+1-\mu|^5} = 0. \quad (2)$$

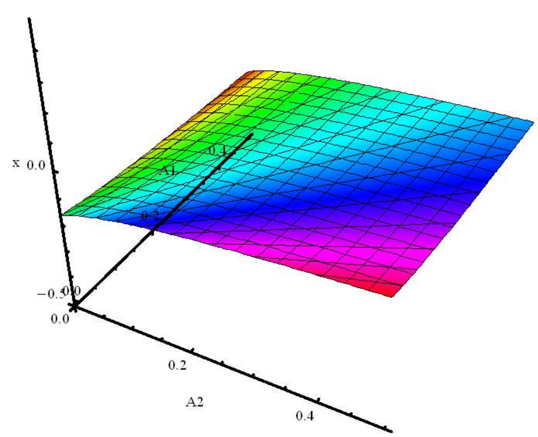
The solutions of eq. (2) have been found to exist within the intervals $(-\infty, -1+\mu)$, $(-1+\mu, \mu)$ and $(\mu, +\infty)$. By solving eq. (2) numerically, each of these intervals contain a real root which correspond to, L_1, L_2 and L_3 respectively. In Table 1, we have shown six cases and their corresponding collinear equilibrium points for all the participating parameters. The first case corresponds to the classical case where $A_1 = A_2 = 0$ and $q_1 = q_2 = 1$. The other five cases have values as shown in the table. The effect of the oblateness and radiation pressure of the binary α-Centuari system on the collinear equilibrium points have also been shown in Figs. 3 and 4 respectively.

Table 1. Collinear Equilibrium points for binary α-Centuari system mass parameter $(\mu = 0.47333)$

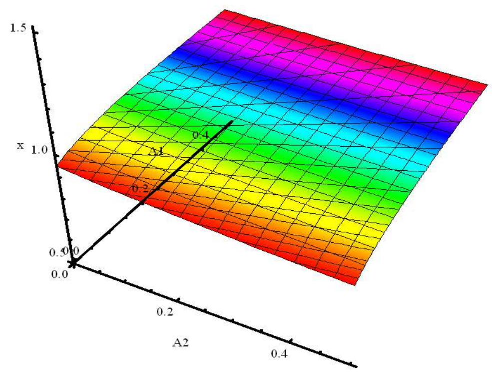
Case	A_1	A_2	q_1	q_2	L_1	L_2	L_3
1	0	0	1	1	-1.20751483	-0.03765997	1.18902157
2	0.01	0.001	0.4	0.1	-0.79423688	-0.16548350	0.95833957
3	0.02	0.002	0.5	0.2	-0.88238502	-0.13509348	1.01122572
4	0.03	0.003	0.6	0.3	-0.94520716	-0.12047798	1.05651685
5	0.04	0.004	0.7	0.4	-0.99536901	-0.11247880	1.09634009
6	0.05	0.005	0.8	0.5	-1.03761166	-0.10790176	1.13198424



(a)

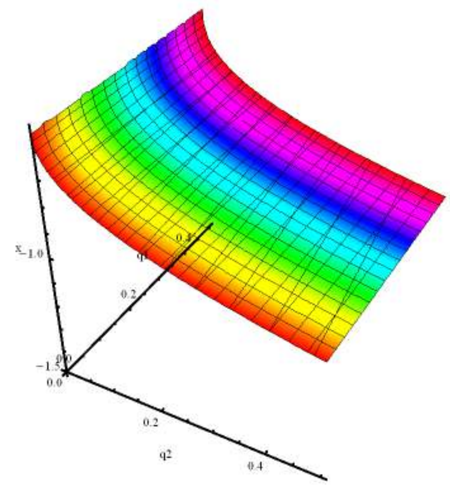


(b)

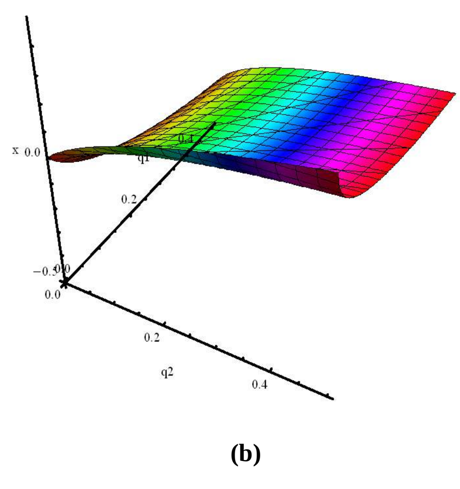


(c)

Fig. 3. The oblateness effects of the binary α-Centuari A system on the collinear equilibrium points L_1 in (a), L_2 in (b) and L_3 in (c), for case 2



(a)



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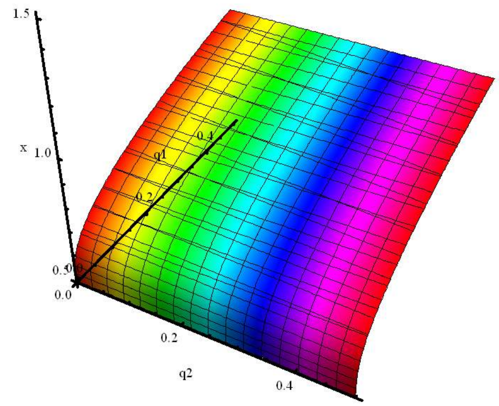


(c)

Fig. 4. The radiation effects of the binary α-Centuari A system on the collinear equilibrium points L_1 in (a), L_2 in (b) and L_3 in (c), for case 2

4 Motion Around the Collinear Equilibrium Points

In order to investigate the motions around the collinear equilibrium points, we obtain a new coordinate system that takes any of L_i, $i = 1, 2, 3$ (the collinear equilibrium points) as the origin with the axes as ϕ and φ parallel to Ox and Oy respectively. Thus, by setting

$$x \to x_{L_i} + \phi \text{ and } y \to \varphi, \tag{3}$$

the equations of motion in eqs. (1) become

$$\begin{aligned} &\ddot{\phi} - 2n\dot{\varphi} = \Omega_{\phi}, \\ &\ddot{\varphi} + 2n\dot{\phi} = \Omega_{\varphi}. \end{aligned} \tag{4}$$

Next, the R.H.S. of eqs. (4) is expanded up to third order terms using the Taylor series expansion and we obtain

$$\begin{aligned} &\Omega_{\phi} = \hbar_1\phi + \hbar_2\phi^2 + \hbar_3\phi^3 + \hbar_4\varphi^2 + \hbar_5\phi\varphi^2, \\ &\Omega_{\varphi} = \bar{\lambda}_1\varphi + \bar{\lambda}_2\phi\varphi + \bar{\lambda}_3\phi^2\varphi + \bar{\lambda}_4\varphi^3, \end{aligned}$$

where

$$\hbar_1 = n^2 + 2q_1(1-\mu)\left(\frac{1}{r_{10}^3} + \frac{3A_1}{r_{10}^5}\right) + 2q_2\mu\left(\frac{1}{r_{20}^3} + \frac{3A_2}{r_{20}^5}\right),$$

$$\hbar_2 = -3(1-\mu)\upsilon_1 q_1\left[\frac{5A_1}{r_{10}^6} + \frac{1}{r_{10}^4}\right]\upsilon_1 + 3\mu q_2\left[\frac{5A_2}{r_{20}^6} + \frac{1}{r_{20}^4}\right]\upsilon_2,$$

$$\hbar_3 = \frac{3(1-\mu)q_1}{2}\left[\frac{5A_1}{r_{10}^6} + \frac{1}{r_{10}^4}\right]\upsilon_1 + \frac{3\mu q_2}{2}\left[\frac{5A_2}{r_{20}^6} + \frac{1}{r_{20}^4}\right]\upsilon_2,$$

$$\hbar_4 = 2(1-\mu)q_1\left(\frac{15A_1}{r_{10}^7} + \frac{2}{r_{10}^5}\right) + 2\mu q_2\left(\frac{15A_2}{r_{20}^7} + \frac{2}{r_{20}^5}\right),$$

$$\hbar_5 = -3(1-\mu)q_1\left(\frac{15A_1}{2r_{10}^7} + \frac{2}{r_{10}^5}\right) - 3\mu q_2\left(\frac{15A_2}{2r_{20}^7} + \frac{2}{r_{20}^5}\right),$$

$$\lambda_1 = n^2 - (1-\mu)q_1\left(\frac{3A_1}{2r_{10}^5} + \frac{1}{r_{10}^3}\right) - \mu q_2\left(\frac{3A_2}{2r_{20}^5} + \frac{1}{r_{20}^3}\right),$$

$$\lambda_2 = 3(1-\mu)q_1\left[\frac{5A_1}{2r_{10}^6} + \frac{1}{r_{10}^4}\right]\upsilon_1 - 3\mu q_2\left[\frac{5A_2}{2r_{20}^6} + \frac{1}{r_{20}^4}\right]\upsilon_2,$$

$$\lambda_3 = -3(1-\mu)q_1\left[\frac{15A_1}{2r_{10}^7} + \frac{2}{r_{10}^5}\right] - 3\mu q_2\left[\frac{15A_2}{2r_{20}^7} + \frac{2}{r_{20}^5}\right],$$

$$\lambda_4 = 3(1-\mu)q_1\left[\frac{5A_1}{4r_{10}^7} + \frac{1}{2r_{10}^5}\right] + 3\mu q_2\left[\frac{5A_2}{4r_{20}^7} + \frac{1}{2r_{20}^5}\right].$$

In order to avoid absolute values for each case, the symbols υ_1 and υ_2 are being used to represent the signs of $r_{10} = |x_1 - \mu|$ and $r_{20} = |x_1 + 1 - \mu|$ at any of the collinear equilibrium points.

We search for periodic solutions represented in the following equations in powers of a parameter ε :

$$\begin{aligned}\phi(\tau) &= \phi_1(\tau)\varepsilon + \phi_2(\tau)\varepsilon^2 + \phi_3(\tau)\varepsilon^3,\\ \varphi(\tau) &= \varphi_1(\tau)\varepsilon + \varphi_2(\tau)\varepsilon^2 + \varphi_3(\tau)\varepsilon^3,\end{aligned} \tag{5}$$

and time is expanded by the expression,

$$t = \kappa\tau,\ \ \kappa = 1 + \rho_2\varepsilon^2 . \tag{6}$$

The symbol ρ_2 has been chosen such that any secular term is eliminated in the course of the computations. Eqns. (4) can now be written as

$$\begin{aligned}\ddot{\phi} - 2n\kappa\dot{\varphi} &= \kappa^2\Omega_\phi,\\ \ddot{\varphi} + 2n\kappa\dot{\phi} &= \kappa^2\Omega_\varphi.\end{aligned} \tag{7}$$

Putting eqs. (5) into (7) and equating the coefficients of $\varepsilon, \varepsilon^2$ and ε^3, we get first order, second order and third order systems respectively, which can be solved successively.

4.1 The First order system

The equations obtained for the first order terms in ε are

$$\begin{aligned} &\phi_1'' - 2n\varphi' - \hbar_1\phi_1 = 0, \\ &\varphi_1'' + 2n\phi' - \lambda\!\!\!^{-}_1\phi_1 = 0. \end{aligned} \tag{8}$$

Defining a differential operator

$$\Gamma(\Im) = \begin{pmatrix} \omega^2 + \hbar_1 & \pm 2n\omega \\ \pm 2n\omega & \omega^2 + \lambda\!\!\!^{-}_1 \end{pmatrix} = \omega^4 + (\hbar_1 + \lambda\!\!\!^{-}_1 - 4n^2)\omega^2 + \hbar_1\lambda\!\!\!^{-}_1 = 0\,. \tag{9}$$

System (8) can also be written as

$$\Gamma(\Im)\begin{pmatrix} \phi_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10}$$

The periodic solution for system (10) is given by

$$\begin{aligned} &\phi_1 = \varpi Cos(\omega\tau) + \vartheta Sin(\omega\tau), \\ &\varphi_1 = \varpi^* Cos(\omega\tau) + \vartheta^* Sin(\omega\tau), \end{aligned}$$

and the period of the periodic orbit is given by $T = \dfrac{2\pi}{\omega}$. We set $\varpi = 1$ and $\vartheta = 0$ so that,

$$\varpi^* = 0 \quad \text{and} \quad \vartheta^* = \frac{-2n\omega}{\lambda\!\!\!^{-}_1 + \omega^2}.$$

Thus, the periodic solution of system (10) become

$$\begin{aligned} &\phi_1(\tau) = Cos(\omega\tau), \\ &\varphi_1(\tau) = \vartheta_1^* Sin(\omega\tau), \end{aligned} \tag{11}$$

where

$$\vartheta_1^* = \frac{-A_{10} - \omega^2}{2n\omega}.$$

4.2 Second order system

This is given by

$$\Gamma(\Im)\begin{pmatrix}\phi_2\\ \varphi_2\end{pmatrix}=\begin{pmatrix}\wp_1(\tau)\\ \wp_2(\tau)\end{pmatrix}, \tag{12}$$

where

$$\begin{aligned}\wp_1(\tau)&=\hbar_2\phi^2+\hbar_4\varphi^2,\\ \wp_2(\tau)&=\lambda_2\phi\varphi.\end{aligned} \tag{13}$$

We substitute eqs. (11) into (12), then eq. (13) become

$$\begin{aligned}\wp_1(\tau)&=\varpi_0+\varpi_1 Cos(2\omega\tau),\\ \wp_2(\tau)&=\vartheta_2^{*}Sin(2\omega\tau),\end{aligned}$$

where

$$\varpi_0=\frac{-\hbar_2+\hbar_3(\vartheta_1^{*})^2}{2\hbar_1},$$

$$\varpi_1=\frac{-\hbar_2(\lambda_1+4\omega^2)+\vartheta_1^{*}[4\lambda_2 n\omega+\hbar_3\vartheta_1^{*}(\lambda_1+4\omega^2)]}{2\hbar_1\lambda_1+8(\hbar_1+\lambda_1-4n^2)\omega^2+32\omega^4},$$

$$\vartheta_2^{*}=\frac{4\hbar_2 n\omega-\vartheta_1^{*}[\hbar_1\lambda_2+4\omega(\hbar_3 n\vartheta_1^{*}+\lambda_2\omega)]}{2\hbar_1\lambda_1+8(\hbar_1+\lambda_1-4n^2)\omega^2+32\omega^4}.$$

Thus, the periodic solution of system (12) is

$$\begin{aligned}\phi_2(\tau)&=\varpi_0+Cos(\omega\tau)+\varpi_1 Cos(2\omega\tau),\\ \varphi_2(\tau)&=\vartheta_1^{*}Sin(\omega\tau)+\vartheta_2^{*}Sin(2\omega\tau).\end{aligned} \tag{14}$$

4.3 Third order system

In this aspect, we have

$$\Gamma(\Im)\begin{pmatrix}\phi_3\\ \varphi_3\end{pmatrix}=\begin{pmatrix}\wp_3(\tau)\\ \wp_4(\tau)\end{pmatrix}, \tag{15}$$

where

$$\wp_3(\tau) = 2n\rho_2\varphi_1' + 2\hbar_3\varphi_1\varphi_2 + 2\hbar_1\rho_2\phi_1 + \hbar_4\varphi_1^2\phi_1 + \hbar_5\phi_1^3 + 2\hbar_2\phi_1\phi_2,$$
$$\wp_4(\tau) = -2n\rho_2\phi_1' + \lambdabar_4\varphi_1^3 + \lambdabar_2\varphi_2\phi_1 + 2\lambdabar_1\rho_2\varphi_1 + \lambdabar_3\phi_1^2\varphi_1 + \lambdabar_2\phi_2\varphi_1,$$

and $\rho_2 = \dfrac{(\Psi_1 + \omega(\Psi_2 - \Psi_3))}{\Psi_4}$,

$$\Psi_1 = -\hbar_1(\vartheta_1^*(\lambdabar_3 - 2\lambdabar_2\varpi_1 + 4\lambdabar_2\varpi_0 + 3\lambdabar_4\vartheta_1^{2*}) + 2\lambdabar_2\vartheta_2^*),$$
$$\Psi_2 = 2n(3\hbar_5 + 4\hbar_2\varpi_1 + 8\hbar_2\varpi_0 + \hbar_4\vartheta_1^{2*} + 4\hbar_3\vartheta_1^*\vartheta_2^*),$$
$$\Psi_3 = (\vartheta_1^*(\lambdabar_3 - 2\lambdabar_2\varpi_1 + 4\lambdabar_2\varpi_0 + 3\lambdabar_4\vartheta_1^{*2}) + 2\lambdabar_2\vartheta_2^*)\omega$$
$$\Psi_4 = (8(\hbar_1(\lambdabar_1\vartheta_1^* - n\omega) + \omega^2(\lambdabar_1\vartheta_1^* + n(-2n\vartheta_1^* + \omega)))).$$

So, working as previously, the periodic solution of system (15) is obtained as

$$\phi_3(\tau) = \varpi_2 Cos(\omega\tau) + \varpi_3 Cos(3\omega\tau),$$
$$\varphi_3(\tau) = \vartheta_3^* Sin(3\omega\tau),$$

where

$\varpi_2 = -\dfrac{\Theta_{21} + \Theta_{22}}{\Theta_{23}}$, with

$$\Theta_{21} = (3\hbar_5 + 4\hbar_2(\varpi_1 + 2\varpi_0) + \vartheta_1^*(\hbar_4\vartheta_1^* + 4\hbar_3\vartheta_2^*))(\lambdabar_1\vartheta_1^* + n\omega),$$
$$\Theta_{22} = -(\lambdabar_3\vartheta_1^* + 3\lambdabar_4\vartheta_1^{3*} + 2\lambdabar_2(-\varpi_1\vartheta_1^* + 2\varpi_0\vartheta_1^* + \vartheta_2^*))(\hbar_1 + n\vartheta_1^*\omega),$$
$$\Theta_{23} = 2(-4n\omega(\hbar_1 + n\vartheta_1^*\omega) + 2(\lambdabar_1\vartheta_1^* + n\omega)(\hbar_1 + \omega^2)),$$

and

$\varpi_3 = -\dfrac{\Theta_{31} + \Theta_{32}}{\Theta_{33}}$, having

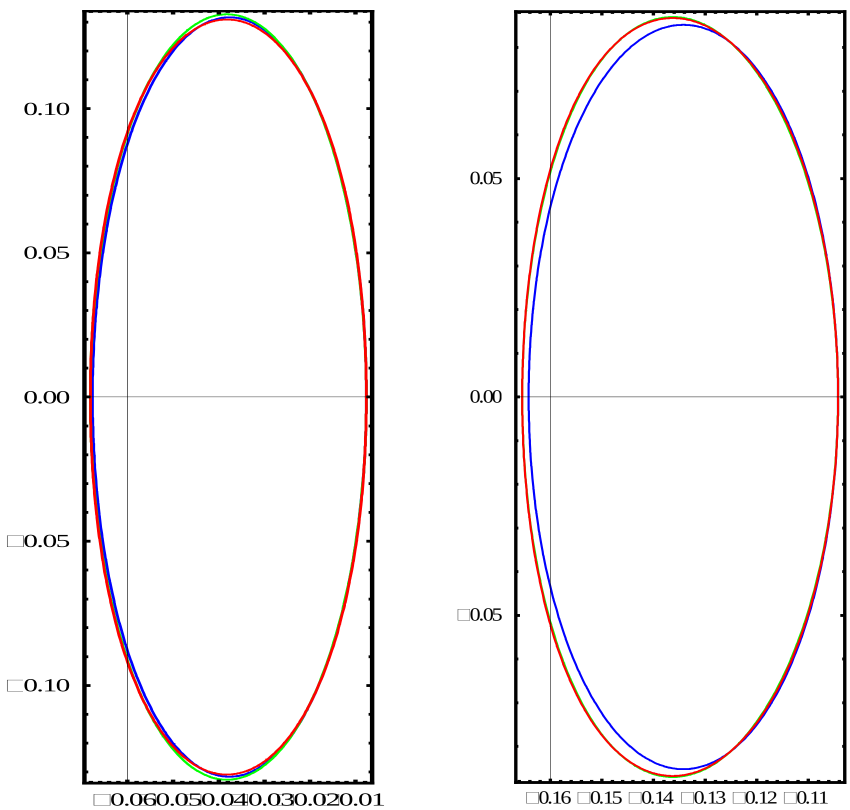
$$\Theta_{31} = \frac{3}{2n}(\lambdabar_3\vartheta_1^* - \lambdabar_4\vartheta_1^{3*} + 2\lambdabar_2(\varpi_1\vartheta_1^* + \vartheta_2^*))\omega,$$
$$\Theta_{32} = -\frac{1}{4}(\hbar_5 + 4\hbar_2\varpi_1 - \vartheta_1^*(\hbar_4\vartheta_1^* + 4\hbar_2\vartheta_2^*))(\lambdabar_1 + 9\omega^2),$$
$$\Theta_{33} = 36n^2\omega^2 - (\hbar_1 + 9\omega^2)(\lambdabar_1 + 9\omega^2),$$

and

$$\vartheta_3^* = -\frac{\Phi_1 + \Phi_2 + \Phi_3}{\Phi_4},$$
$$\Phi_1 = \hbar_1(\hat{\lambda}_3\vartheta_1^* - \hat{\lambda}_4\vartheta_1^{3*} + 2\hat{\lambda}_2(\varpi_1\vartheta_1^* + \vartheta_2^*),$$
$$\Phi_2 = -6n(\hbar_5 + 4\hbar_2\varpi_1 - \vartheta_1^*(\hbar_4\vartheta_1^* + 4\hbar_3\vartheta_2^*))\omega,$$
$$\Phi_3 = 9(\hat{\lambda}_3\vartheta_1^* - \hat{\lambda}_4\vartheta_1^{3*} + 2\hat{\lambda}_2(\varpi_1\vartheta_1^* + \vartheta_2^*))\omega^2,$$
$$\Phi_4 = 4(\hbar_1\hat{\lambda}_1 + 9(\hbar_1 + \hat{\lambda}_1 - 4n^2)\omega^2 + 81\omega^4.$$

Therefore, the third order approximation of periodic solution under oblate binary α -Centuari A system around the collinear equibrium points as a function of parameter ε is being obtained as

$$\begin{aligned}
\phi(\tau) &= [cos(\omega\tau)]\varepsilon + [\varpi_0 + \varpi_1 cos(2\omega\tau)]\varepsilon^2 + [\varpi_2 cos(\omega\tau) + \varpi_3 cos(3\omega\tau)]\varepsilon^3,\\
\varphi(\tau) &= [\vartheta_1^* sin(\omega\tau)]\varepsilon + [\vartheta_2^* sin(2\omega\tau)]\varepsilon^2 + [\vartheta_3^* sin(3\omega\tau)]\varepsilon^3,\\
\dot{\phi}(\tau) &= -\omega\sin(\omega\tau)\varepsilon - 2\omega\varpi_1\sin(2\omega\tau)\varepsilon^2 - [\omega\varpi_2\sin(\omega\tau) + 3\varpi_3\sin(3\omega\tau)]\varepsilon^3,\\
\dot{\varphi}(\tau) &= \omega\vartheta_1^*\cos(\omega\tau)\varepsilon + 2\omega\vartheta_2^*\cos(2\omega\tau)\varepsilon^2 + 3\omega\vartheta_3^*\cos(3\omega\tau)\varepsilon^3.
\end{aligned} \tag{16}$$



(a) **(b)**

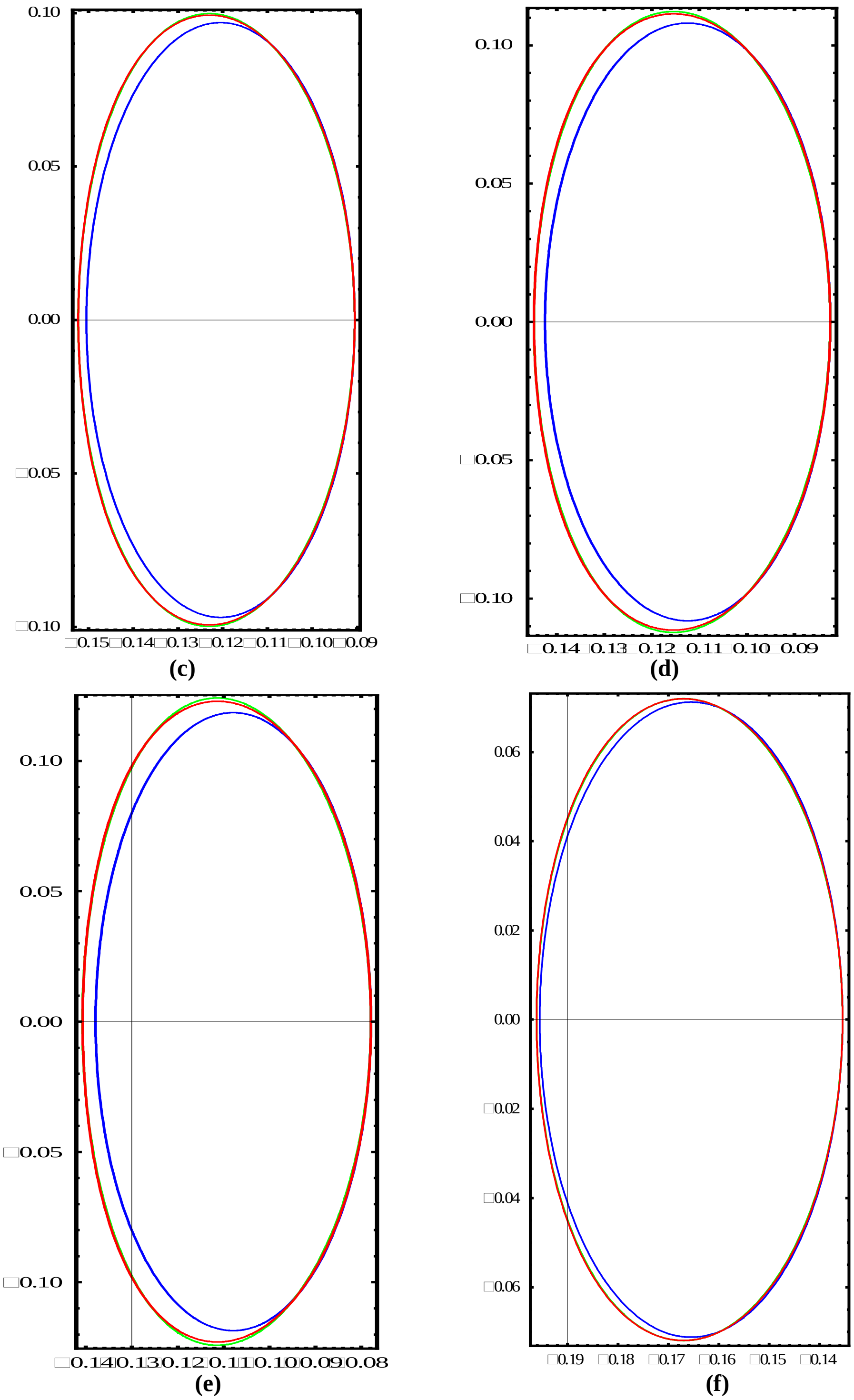


Fig. 5. In (a), (b), (c), (d), (e) and (f) [which also correspond to figures for cases 1,2,3,4,5 and 6] we show the starting orbits where in each frame, the orbits in colour blue, green and red correspond to the first, second and third order systems respectively

5 Numerical Results

In the analysis presented above, we have obtained a third order approximation of periodic solution with the aid of the Lindsted-Poincaré method around the collinear equilibrium points by taking the primaries as oblate and radiating bodies in the restricted three-body problem. The mass parameter $\mu = 0.47333$ is obtained from the binary α-Centuari system. As such, the initial conditions $\left(x_0, \dot{y}_0, T \right)$ as shown in Tables 2, 3 and 4 have been obtained by substituting the data presented in Table 1 in eqs. (16) for small values of the orbital parameter ε in all the cases (1 to 6) at $\tau = 0$. Also, in Tables 2, 3 and 4, the fifth column and sixth column give the Jacobi Constant and the positive imaginary root $(+\omega i)$ to the characteristic equation of system (9), respectively.

In Fig. 4, we show in (a), (b), (c), (d), (e) and (f) [which also correspond to figures for cases 2,3,4,5 and 6], the starting orbits where in each frame, the orbits in colour blue, green and red correspond to the first, second and third order systems respectively. It can be seen that these starting orbits are all oval in shape [14].

Table 2. The initial conditions for the Lyapunov orbits around the collinear equilibrium point L_1 for binary α-Centuari system mass parameter $\mu = 0.47333$

Case	x_0	$\dot{y}_0$	T	C	ω
1	-1.17772296	-0.15844398	5.26996513	3.47518463	1.19274305
2	-0.76451797	-0.13404904	3.32511205	1.33092657	1.89037668
3	-0.85296994	-0.14128336	3.76434523	1.73135624	1.66980282
4	-0.91603690	-0.15064417	4.03701109	2.07085351	1.55702180
5	-0.96642262	-0.16040252	4.23423349	2.37809889	1.48449874
6	-1.00888380	-0.17018986	4.38698369	2.66454003	1.43281004

Table 3. The initial conditions for the Lyapunov orbits around the collinear equilibrium point L_2 for binary α-Centuari system mass parameter $\mu = 0.47333$

Case	x_0	$\dot{y}_0$	T	C	ω
1	-0.00786810	-0.50240508	1.66199403	3.99865977	3.78203180
2	-0.13576459	-0.09924098	4.49137120	0.95857994	1.39950897
3	-0.10567840	-0.17260044	3.08133233	1.39453273	2.03993390
4	-0.09130772	-0.24469574	2.48534026	1.83036201	2.52911619
5	-0.08353241	-0.31972218	2.12428715	2.27039126	2.95897580
6	-0.07917390	-0.39856305	1.87327991	2.71609925	3.35545918

Table 4. The initial conditions for the Lyapunov orbits around the collinear equilibrium point L_3 for binary α-Centuari system mass parameter $\mu = 0.47333$

Case	x_0	$\dot{y}_0$	T	C	ω
1	1.21881344	-0.14860729	5.61879746	3.43731738	1.11869387
2	0.98805848	-0.10827148	4.11676337	1.88451257	1.52685829
3	1.04064080	-0.12551658	4.23720387	2.19245704	1.48345807
4	1.08568711	-0.14022498	4.33697474	2.48247540	1.44933154
5	1.12528648	-0.15369264	4.41909062	2.75927019	1.42239995
6	1.16071210	-0.16640484	4.48676943	3.02584346	1.40094435

6 Conclusions

In this study, we have obtained a third order analytic approximation of Lyapunov orbits around the collinear equilibrium points in the R3BP by using the Lindstedt-Poincaré method. We modelled the primaries in the binary α-Centuari system where the primary body is α-Centuari A and the secondary body is α-Centuari B. The infinitesimal body is taken to be a possible exoplanet moving in the plane of motion of the binary system.

Also, we numerically determined the positions of the collinear equilibrium points and showed the effects of the parameters concerned on these points. It can be seen that with each increase in the radiation pressure and oblateness parameters, the first collinear point L_1 moves away from the origin and closer to the position of that of the classical case and also closer to the secondary body, while the second collinear equilibrium point (inner collinear point) L_2 moves toward the origin and closer to that of the classical case. The third collinear equilibrium point L_3 moves further away from the primary body and approaches the classical case.

The initial conditions or starting orbits obtained in all the cases are shown in both tabular and graphical forms in Tables 2, 3 and 4 and Figs. 4 respectively. The orbits are oval in shape. These results can be used to continue to families of periodic orbits and can be combined with those of the spatial orbits as well.

Acknowledgements

The authors are grateful to Asst. Prof. V. S. Kalantonis for his invaluable support both in suggestions and materials.

Competing Interests

Authors have declared that no competing interests exist.

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