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On Properties Related To ∗–Reversible Rings

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Authors' contributions

This work was carried out in collaboration between both authors. Author NAAJ designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors WMF and NAAJ managed the analyses of the study. Author WMF managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, a class of ∗-rings which is a generalization of ∗–reversible rings is introduced. A ring with involution ∗ is called central ∗–reversible if for $a, b \in R$, whenever $ab = 0$, b^*a is central in R. Since every ∗–reversible ring is central ∗–reversible, sufficient conditions for central ∗–reversible rings to be ∗–reversible is studied. We show that some results of ∗–reversible rings can be extended to central ∗–reversible ring. For an Armendariz ring R, we prove that R is central ∗–reversible if and only if the polynomial ring $R[x]$ is central ∗–reversible if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central ∗–reversible.

Keywords: ∗-reversible rings; weakly ∗–reversible rings; central reversible rings; central ∗–reversible rings.

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1 Introduction

Throughout this note we assume that rings are associative with identity unless otherwise stated. A ring R is *reduced* if it has no non zero nilpotent elements. A ring is called *central reduced* [1] if every nilpotent element of R is central. A ring R is called *semicommutative* if for all $a\,,b \in R, ab = 0$ implies $aRb = 0$ [2]. According to Lambek [3], a ring R is *symmetric* if for any $a\,,b\,,c \in R, abc = 0$ implies $acb = 0$ if and only if $abc = 0$ implies $bac = 0$. R is called *reversible* ring if $ab = 0 \leftrightarrow\ ba = 0$, for any $a\,,b \in R$ [4]. *Central reversible* ring R is defined by Kose, et al. in [5] as follows: If for any $a\,,b \in R\,,ab = 0$ implies ba is central in R. Reduced rings, central reduced rings, symmetric rings and reversible rings are central reversible. An additive mapping $*: R \to R$ is called *an involution* if $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a\,,b \in R$. A ring equipped with an involution is called *a ring with involution* or $*$*–ring*. We say that an involution $*$ of a ring R is a *semiproper* involution if for any $a \in R, aRa^* = 0$ implies $a = 0$ [6]. Recently, the notion of reversibility is defined for a $*$-ring [7]. A ring R with an involution $*$ is called $*$*–reversible* if for any $a\,,b \in R, ab = 0$ implies $b^*a = 0$. A $*$–reversible ring is symmetric, reversible and semicommutative ring. A ring R with involution $*$ is called $*$*–symmetric* if for any elements $a\,,b\,,c \in R, abc = 0$ implies $acb^* = 0$ [8]. It is clear that $*$–symmetric ring with unity is $*$–reversible. For $*$–reversible ring R, it is proven that R is symmetric if and only if R is $*$–symmetric [8]. A ring R is called *right (left) principally quasi-Baer* [9] if the right (left) annihilator of a principal right (left) ideal of R is generated by an idempotent. Finally, a ring R is called *right (left) principally projective* if the right (left) annihilator of an element of R is generated by idempotent [10]. Throughout this paper, we use $Z(R), N(R)$ and $P(R)$ to denote the center of a ring R, the set of all nilpotent elements in R and the prime radical, respectively. We write $R[x]$ and $R[x\,,x^{-1}]$ for the polynomial ring and the Laurent polynomial ring, respectively.

2 Central $*$–Reversible Rings

In this section we introduce a class of rings, called central $*$–reversible rings, which is a generalization of $*$–reversible rings.

2.1. Definition: A ring R with an involution * is called central *–reversible if whenever ab=0 for a, b∈R,b^* a is central in R.

Cleary, $*$-reversible rings are central $*$–reversible. We supply an example to show that all central $*$–reversible rings need not be $*$–reversible. We show that central $*$–reversible rings are weakly $*$–reversible rings. We prove that central $*$–reversible rings are abelian and there exists an abelian ring but not central $*$–reversible. We prove that every central $*$–reversible ring is central $*$–semicommutative and 2-primal. Moreover, we prove that if R is reduced and central $*$–reversible ring, then the trivial extension $T(R,R)$ is central $*$–reversible. For an Armendariz ring, we prove that R is central $*$–reversible if and only if the polynomial ring $R[x]$ is central $*$–reversible if and only if the Laurent polynomial $R[x\,,x^{-1}]$ is central $*$–reversible. Finally, the Dorroh extension of R is central *–reversible if and only if a ring R is central $*$–reversibl.

2.2. Example: Let R be a commutative ring and consider the ring

$S = \left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a\,,\ b\,,\ c \in R\right\}$. Let $*$ be an involution on S defined by

$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$. Let $= \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}$, B $= \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix} \in S$ with $AB = 0$.

Then

$$B^*A = \begin{pmatrix} 0 & c_2 & b_2 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_2c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in Z(S).$$

Therefore, S is central $*$-reversible.

Clearly, every $*$-reversible ring is central $*$-reversible. In the next example we will see the converse is not true in general.

2.3. Example: Let $R = Z_{10} \oplus Z_{10}$ which is a commutative ring under usual multiplication. Define the exchange involution $*$ on R by $(a\,,b)^* = (b,a)$, for all $(a\,,b) \in R$. It is clear that R is central $*$–reversible. Now, let $a = (5\,,0), b = (4\,,3)$; then we see that $ab = 0$ while $b^*a = (3\,,4)(5\,,0) = (5\,,0) \neq 0$. Hence R is not $*$–reversible.

Our next study is to find conditions under which a central $*$–reversible ring is $*$–reversible.

2.4. Proposition: If R is a central $*$-reversible ring, then R is $*$-reversible if R satisfies any of the following conditions.

$1-$ R is a ring with semiproper involution $*$.
$2-$ R is a right (left) principally projective ring.
$3-$ R is a right (left) principally quasi-Baer ring.

Proof. First statement is clear. Conversely, assume that R is a central $*$-reversible ring and $a\,,\, b \in R$ with $ab = 0$. Now consider the following cases.

$1-$ Let R be a ring with semiproper involution $*$. Since b^*a is central, $b^*aR(b^*a)^* = b^*aRa^*b = Ra^*b^*ab = 0$ and so $b^*a = 0$. Thus R is $*$-reversible.
$2-$ Let R be a right principally projective ring. Then there exists a central idempotent $e \in R$ such that $r_R(a) = eR$. Hence $ae = 0$. Since $b \in r_R(a) = eR$, we have $b = eb$. It follows that $b^*a = (eb)^*a = b^*ea = b^*ae = 0$. Thus R is $*$-reversible. A similar proof may be given for left principally projective rings.
$3-$ Same as the proof of (2).

2.5. Corollary: Let R be a ring with involution $*$. If R is central $*$–reversible, then the conditions below are equivalent.

(1) R is a right (left) principally quasi-Baer ring.
(2) R is a right (left) principally projective ring.

Next we show that central $*$–reversible rings are closed under finite direct sums.

2.6. Proposition: Let $\{R_i\}_{i\in I}$ be a class of rings for a finite index set I. Then R_i is central $*$–reversible for all $i \in I$ if and only if $\oplus_{i\in I} R_i$ is central $*$–reversible.

Proof. The necessity follows from definitions. The sufficiency is clear since a subring of central $*$–reversible ring is central $*$–reversible.

The following result is a direct consequence of Proposition 2.6.

2.7. Corollary: Let R be a $*$–ring. Then eR and $(1-e)R$ are central $*$–reversible for some central idempotent e in R if and only if R is central $*$–reversible.

2.8. Remark: If R is $*$–reversible ring without unity, then R is $*$–symmetric. Suppose that $abc = 0$, then $bca = 0$. Hence $a^*bc = c^*a^*b = (c^*a^*)^*b^* = acb^* = 0$. Therefore R is $*$–symmetric.

2.9. Lemma: Let R be $*$–reversible ring. If $ab \in N(R)$ for $a , b \in R$, then $b^*a \in N(R)$.

Proof. Let R be $*$–reversible ring. Assume that $ab \in N(R)$ for $a , b \in R$. Then there exists a positive integer n such that $(ab)^n = 0$. By above remark, R is $*$–symmetric. It follows that

$$b^*a(ab)^{n-1} = 0 \Longrightarrow abb^*a(ab)^{n-2} = 0.$$
$$\Longrightarrow b^*ab^*a(ab)^{n-2} = 0.$$
$$\Longrightarrow (b^*a)^2(ab)^{n-2} = 0.$$
$$\Longrightarrow (b^*a)^3(ab)^{n-3} = 0.$$

Using a similar method we get $(b^*a)^{n-1}ab = ab(b^*a)^{n-1} = b^*a(b^*a)^{n-1} = (b^*a)^n = 0$. Therefore, $b^*a \in N(R)$.

2.10. Lemma: If R is central $*$–reversible ring, then it is abelian.

Proof. Let e be an idempotent of R. For any $r \in R$, $(re - ere)(e - 1) = 0$ implies $(e - 1)^*(re - ere) = ere - re$ is central. Commuting $ere - re$ by e we have $ere - re = 0$. Similarly for any $r \in R, (e - 1er - ere=0$ implies $ere - er=0$. Therefore R is abelian.

Every abelian ring need not be central $*$–reversible for some involution $*$ as the following example shows.

2.11. Example: Consider the ring

$R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a , b , c , d \in \mathbb{Z}, a \equiv d (mod 2), b \equiv c \equiv 0 (mod 2)\right\}$. Since 0 and the identity matrices are the only idempotents of R, R is abelian ring. Define an involution $*$ on R by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. On the other hand, consider $A = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ with $AB = 0$. But B^*A is not central for $C = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix} \in R$. Hence R is not central $*$–reversible.

Recall that a ring R is called *directly finite* whenever $a, b \in R, ab = 1$ implies $ba = 1$. Then we have the following.

2.12. Corollary: Every central $*$–reversible ring is directly finite.

Recall that a ring R is called *unit-central* [11], if all unit elements are central in R. It is proven that every unit-central ring is abelian.

2.13. Lemma: Let R be a unit central and $*$–reversible ring. If I is a nil ideal of R, then R/I is central $*$–reversible.

Proof. Let $a , b \in R$ with $(a + I)(b + I) = 0 + I$. Then $ab + I = I$ and so $ab \in I$. Therefore there exists a positive integer n such that $(ab)^n = 0$. Hence $(b^*a)^n = 0$. It follows that $b^*a \in N(R) \subset Z(R)$ [11]. Thus $rb^*a = b^*ar$ for any $r \in R$. Therefore $(b^* + I)(a + I)$ is central in R/I.

The given example proves that for if R is a ring with involution and an ideal I, if R/I is central $*$–reversible, then R need not be central $*$–reversible.

2.14. Example: Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, where $\mathbb{Z}$ the set of integer. Consider the ideal $I = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ of R. Then $R/I = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} + I, a \in \mathbb{Z} \right\}$ is central $*$–reversible. Let $*$ be an involution on R defined by $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^* = \begin{bmatrix} a & -b \\ 0 & c \end{bmatrix}$. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in R$, we have $AB = 0$ but $B^*A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ is not central in R. Therefore R is not central $*$–reversible.

2.15. Lemma: Let R be a $*$–ring. Then R is domain if and only if it is a prime and central $*$–.

Proof. Let $a, b \in R$ with $ab = 0$. Then $xab = 0$ for any $x \in R$ and so b^*xa is central. Then $tb^*xab = 0 = b^*xatb$ for any $t \in R$. Since R is prime, $b^* = 0$ or $atb = 0$ and so $b = 0$ or $a = 0$. The rest is clear.

A ring R is called α–semicommutative if whenever $ab = 0$ for $a, b \in R, aR\alpha(b) = 0$, where $\alpha: R \to R$ is an endomorphism [12]. By replacing the endomorphism α by the involution $*$ which is an anti-automorphism of R of order two, we have a ring R with involution $*$ is said to be $*$–semicommutative, if whenever $ab = 0$ for $a, b \in R$, $aRb^* = 0$ and is called central $*$-semicommutative if $ab = 0$ implies aRb^* is central for $a, b \in R$. A ring R with involution $*$ is called $*$–rigid, if for any $a \in R, aa^* = 0$, then $a = 0$ [8], while the ring R is said to be central $*$–rigid if for any $a \in R, aa^* = 0$ implies a is central.

2.16. Theorem: Let R be a right principally projective ring. Then the following are equivalent.

1 – R is reduced
2 – R is central $*$-rigid.
3 – R is central $*$–reversible.
4 – R is central $*$-semicommutative.
5 – R is abelian.

Proof. Note first that if R is a right principally projective ring, then every idempotent is central.

$(1) \Longrightarrow (2)$ Let $a \in R$ with $aa^* = 0$. Then we have $(a^*ra)^2 = 0$ and so $a^*ra = 0$ since R is reduced. We have $a \in r_R(a^*r) = eR$ for some $e^2 = e \in R$. So $a = ea$ and $a^*re = 0$. If $r = 1, a^*e = 0$ and $ea^* = 0$. Since $a^* \in r_R(a), a^* = ea^* = 0$. Hence $a = 0$ and so central.

$(2) \Longrightarrow (3)$ Let $a, b \in R$ with $ab = 0$. Then $b \in r_R(a) = eR$ for some $e^2 = e \in R$. So $b = eb$ and $ae = 0$. On the other hand $b^*a^* = 0$. Then $a^* = ea^*$ and $b^*e = 0$. We have $b^*a(b^*a)^* = (eb)^*aa^*b = b^*eaa^*b = 0$. Since R is central $*$–rigid, b^*a is central. Hence R is central $*$–reversible.

$(3) \Longrightarrow (4)$ Let $a, b \in R$ with $ab = 0$, then $b^*a^* = 0$. For all $x \in R, xb^*a^* = 0$. Since R is central $*$–reversible, axb^* is central. Hence R is central $*$–semicommutative.

$(4) \Longrightarrow (5)$and $(5) \Longrightarrow (1)$ Clear.

2.17. Corollary [13, Corollary 2.21]: Let R be a ring. Then the following are equivalent.

1 – R is central reduced.
2 – R is abelian and for any idempotent $\in$ R , eR and $(1 - e)$R are central reduced.

2.18. Lemma: Let R be a ring with involution $*$. If R is central $*$–reversible, then R is central reduced.

Proof. Let R be central $*$–reversible ring. Then eR and $(1 - e)R$ are central $*$–reversible by corollary 2.7 and right principally projective rings. By theorem 2.16, eR and $(1 - e)R$ are reduced. By corollary 2.17, R is central reduced.

Recall that the ring R is called *2-primal* if $P(R) = N(R)$.

2.19. Theorem: If R is a central $*$–reversible ring, then it is 2-primal. The converse holds for rings with semiproper involution $*$.

Proof. Let R be a central $*$–reversible ring. We have $P(R) \subseteq N(R)$. To prove the converse , let $a \in N(R)$ with $a^n = 0$ for some positive integer n. Suppose that $a \notin P$ for a prime ideal P. Since R is central reduced, a is central. For any $r_{n-1}, r_{n-2}, \dots, r_2, r_1 \in R$, we have $ar_{n-1}ar_{n-2}a \dots ar_2ar_1a = r_{n-1}r_{n-2} \dots r_2r_1a^n = 0$. For all prime ideals K, we have $aR(ar_{n-2}a \dots ar_2ar_1a) \subseteq K$. Since $a \notin P, ar_{n-2}a \dots ar_2ar_1a \in K$ for all prime ideals K and $r_{n-2}, \dots, r_2, r_1 \in R$. Hence $aR(ar_{n-3}a \dots ar_2ar_1a) \subseteq K$ for all prime ideals K and $r_{n-3}, \dots, r_2, r_1 \in R$. Using a similar reasoning, since $a \notin P$, $aR(ar_{n-4}a \dots ar_2ar_1a) \subseteq K$ for all prime ideals K and for all $r_{n-4}, \dots, r_2, r_1 \in R$ implies $ar_{n-4}a \dots ar_2ar_1a \in K$ for all prime ideals K and for all $r_{n-4}, \dots, r_2, r_1 \in R$. By going downward induction, we may reach $aRa \subseteq K$ for all prime ideals K. Hence $a \in K$ for all prime ideals K, a contradiction. Thus if a is nilpotent, then $a \in P(R)$ and so $N(R) \subseteq P(R)$. Conversely, let R be a 2-primal ring with semiproper involution $*$. Then $P(R) = 0$ and so $N(R) = 0$. Hence R is reduced. Let $ab = 0$. Then $abb^* = bb^*a = a^*bb^*a = 0$. Then we have $[(b^*a)r(a^*b)]^2 = b^*ara^*bb^*ara^*b = 0$. Since R is reduced, $(b^*a)r(a^*b) = 0$. We have $b^*a = 0$ and so b^*a is central. Hence R is central $*$–reversible. This completes the proof.

A ring R with involution $*$ is said to be *weakly $*$–reversible*, if for all $a , b , r \in R$ such that $ab = 0, Rb^*ra$ is a nil left ideal of R.

2.20. Theorem: Let R be a ring with involution $*$. Consider the following conditions.

(1) R is $*$–reversible.
(2) R is central $*$–reversible.
(3) R is weakly $*$–reversible.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Let $a , b \in R$ with $ab = 0$. Then for all $y \in R, yab = 0$. Since R is central $*$–reversible, b^*ya is central. Then we have $rbrb^*yar = 0$ for all $r , y \in R$. This implies that $(b)(b^*ya) = 0$. Then $(b)(b^*ya) \subseteq P(R)$. Since every central $*$–reversible is 2-primal, $b \in N(R)$ or $b^*ya \in N(R)$. If $b \in N(R)$, then there exists a positive integer n such that $b^n = 0$. Then we have $(rb^*ya)^n = r(b^*)^n yaya \dots r = 0$ and so Rb^*ya is a nil left ideal of R. If $b^*ya \in N(R)$, then there exists a positive intger m such that $(b^*ya)^m = 0$. Then we have $(rb^*ya)^m = r(b^*ya)^m \dots r = 0$ and so Rb^*ya is a nil left ideal of R. Therefore R is weakly $*$–reversible.

2.21. Lemma: Let R be weakly $*$-reversible ring. If R/I is a central $*$–reversible ring with a reduced ideal I, then R is central $*$–reversible.

Proof. Let $a , b \in R$ with $ab = 0$. Since R is weakly $*$-reversible, Rb^*ra is nil left ideal of R. Then $Ib^*ra \subseteq Rb^*ra\ (I \subseteq R)$. If $r = 1$, then $Ib^*a \subseteq Rb^*ra \in N(R)$. So there exists a positive integer n such that $(Ib^*a)^n = 0$. Since I is reduced, $Ib^*a = 0$. Now, let R/I be central $*$–reversible ring. Let $a , b \in R$ with $ab = 0$. Since $(a + I)(b + I) = 0, (b^* + I)(a + I)$ is central in R/I. It follows that $b^*ar - rb^*a \in I$ for any $r \in R$. Then $I(b^*ar - rb^*a) = 0$. Hence we have $(b^*ar - rb^*a)^2 = 0$. Since I is reduced, $b^*ar = rb^*a$ and so R is central $*$–reversible.

Let R be a ring and M an (R, R)–bimodule. Recall that the trivial extension of R by M is defined to be ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This ring is isomorphic to the ring $\left\{\begin{bmatrix} r & m \\ 0 & r \end{bmatrix} : r \in R, m \in M\right\}$ with the usual matrix operations and isomorphic to $R[x]/(x^2)$, where (x^2) is the ideal generated by x^2. An induced involution on the trivial extension $T(R, R)$ of R with involution $*$ is given by $\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}^* = \begin{pmatrix} r^* & s^* \\ 0 & r^* \end{pmatrix}$ [8].

2.22. Proposition: If R with involution $*$ is reduced and central $*$–reversible ring, then $T(R,R)$ is central $*$–reversible.

Proof. Let $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \in T(R,R)$ with $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = 0$. Then

$$ac = 0$$

$$ad + bc = 0 \implies ad = -bc$$

Since R is central $*$–reversible, c^*a is central. Hence $(ad)^3 = (-bc)(ad)(-bc) = b(ca)dbc = bdbcac = 0$. Since R is reduced, $ad = -bc = 0$ which implies d^*a, c^*b are central in R. Therefore $\begin{bmatrix} c^* & d^* \\ 0 & c^* \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \in Z(T(R\,,R))$.

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let $S^{-1}R$ be the localization of R at S. Define an involution $*$ on R by $(s^{-1}r)^* = s^{-1}r^*$. Then we have the following proposition.

2.23. Proposition: A ring R with involution $*$ is central $*$–reversible if and only if $S^{-1}R$ is central $*$–reversible.

Proof. Let R be a central $*$–reversible ring and $a/r\,, b/s \in S^{-1}R$ where $,b \in R$, $r\,, s \in S$ with $(a/r)(b/s) = 0$. Since $(a/r)(b/s) = ab/rs = 0$ we have $ab = 0$. By hypothesis b^*a is central, so $(b^*/s)(a/r)(c/t) = b^*ac/srt = cb^*a/tsr = (c/t)(b^*/s)(a/r)$ for every $c/t \in S^{-1}R$, where $c \in R$ and $t \in S$. Therefore $S^{-1}R$ is central $*$–reversible. Conversely, assume that $S^{-1}R$ is a central $*$–reversible ring. Since R may be embedded in $S^{-1}R$, the rest is clear.

2.24. Corollary: Let R be $*$-ring. Then $R[x]$ is central $*$-reversible if and only if $R[x, x^{-1}]$ is central $*$–reversible.

Proof. Consider the subset $S = \{1\,, x\,, x^2, x^3, \ldots\}$ of $R[x]$ consisting of central regular elements. Then it follows from Proposition 2.23.

Let R be a $*$-ring and $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$. Rege and Chhawchharia [14] introduce the notion of an Armendariz ring, that is, a ring R is called *Armendariz*, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j. Define an involution $*$ by $f^*(x) = \sum_{i=0}^{n} a_i^* x^i$, for every polynomial $f(x) \in R[x]$ [7].

2.25. Theorem: Let R with involution $*$ be an Armendariz ring. Then the following statements are equivalent.

1 – R is central $*$–reversible.
2 - $R[x]$ is central $*$–reversible.
3 - $R[x, x^{-1}]$ is central $*$–reversible.

Proof. (1) $\implies$ (2) Let $f(x) = \sum_{i=0}^{n} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ with $f(x)g(x) = 0$. Since R is Armendariz, $a_i b_j = 0$ for each i and j. But R is central $*$–reversible so $b_j^* a_i$ is central for each i and j. It follows that $g^*(x)f(x)$ is central in $R[x]$. Therefore $R[x]$ is central $*$–reversible.

(2) $\implies$ (1) Let $a_i, b_j \in R$ with $a_i b_j = 0$ for each i and j. Then $f(x)g(x) = 0$ where $f(x), g(x) \in R[x]$. Since $R[x]$ is central $*$–reversible, $g^*(x)f(x) \in Z(R[x])$ and so $b_j^* a_i \in Z(R)$. Therefore R is central $*$–reversible.

(2) $\Leftrightarrow$ (3) It follows from Corollary 2.24.

A ring R is called *nil-Armendariz* [15] if whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in nil(R)[x]$ then $ab \in nil(R)$ for all $a \in coef(f(x))$ and $b \in coef(g(x))$.

2.26. Proposition: If R with involution $*$ is central $*$–reversible, then R is nil-Armendariz.

Proof. If R is central $*$–reversible, then it is 2-primal by theorem 2.19 and so $N(R)$ is an ideal of R. [15, Proposition 2.1] states that in a ring in which the set of all nilpotent elements forms an ideal, then the ring is nil-Armendariz.

The *Dorroh extension* $D(R,Z) = \{(r\ ,n): r \in R, n \in Z\}$ of a ring R is a ring with operations $(r_1,n_1) + (r_2,n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1,n_1)(r_2,n_2) = (r_1r_2 + n_1r_2 + n_2r_1, n_1n_2)$. Obviously R is isomorphic to the ideal $\{(r,0): r \in R\}$ of $D(R,Z)$. If the algebra R adheres to an involution $*$, then an induced involution $*_D$ on D is $(r,n)^{*_D} = (r^*\ ,n)$ for every $(r\ ,n) \in D$ [7]. Then we have the following.

2.27. Proposition: A ring R with involution $*$ is central $*$–reversible if and only if the Dorroh extension $D(R,Z)$ of R is central $*$–reversible.

Proof. The sufficiency is clear. For necessity, let $(r_1,n_1)\ ,(r_2,n_2) \in D(R,Z)$ with $(r_1,n_1)(r_2,n_2) = 0$. Then $n_1n_2 = 0$. Assume that $n_1 = 0$. Since R is central $*$-reversible, $(r_2^* + n_2)r_1$ is central in R and so $(r_2^*, n_2)(r_1,n_1)$ is central in $D(R\ ,Z)$. Hence $D(R\ ,Z)$ is central $*$–reversible. A similar proof may be given for $n_2 = 0$.

3 Conclusion

In this paper the study introduced central $*$-reversible ring (Definition 2.1), which generalized the concept of $*$-reversible ring, published in [7]. Moreover it established a number of properties of this generalization. The connection between central $*$-reversible and other rings was also investigated (Theorem 2.16). Finally it proved that if R is Armendariz ring then, R is central $*$-reversible if and only if R[x] is central $*$-reversible if and only if R $[x, x^{-1}]$ is central $*$-reversible (Theorem 2.25).

Competing Interests

Authors have declared that no competing interests exist.

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