Article

Generalized fractional differential ring

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Abstract: There are many possible definitions of derivatives, here we present some and present one that we have called generalized that allows us to put some of the others as a particular case of this but, what interests us is to determine that there is an infinite number of possible definitions of fractional derivatives, all are correct as differential operators each of them must be properly defined its algebra. We introduce a generalized version of fractional derivative that extends the existing ones in the literature. To those extensions it is associated a differentiable operator and a differential ring and applications that shows the advantages of the generalization. We also review the different definitions of fractional derivatives and it is shown how the generalized version contains the previous ones as a particular cases.

Keywords: Generalized fractional derivative; Fractional differential ring; Fractional differential equation.

MSC: Missing

1. Introduction and Preliminaries

$\boxed{\text{F}}$ ractional derivative was defined for responding to a question 'what does it mean $\frac{d^\alpha f}{dt^\alpha}$ if $\alpha = \frac{1}{2}$' in 1695. Following that, finding the right definition of fractional derivative has attracted significant attention of researcher and in the last few years it has seen significantly progress in mathematical and non-mathematical journals (see [1–10]). In fact, there are articles which in few months have gained hundreds of citations. In particular in past three years several definitions of fractional derivative have been proposed (see [3,4,10–23]). Since some of previous definitions do not satisfy the classical formulas of the usual derivative, it has been proposed an ad hoc algebra associated to each definition. To unify that diversity, we propose a version of fractional derivative that has the advantages that generalized the already existing in the literature and where the different algebras are unified under the notion of fractional differential ring.

The present paper is organized as follows: In the Section 2 we give the previous definitions of fractional derivative and our generalized fractional derivative (GFD) definition, in the Section 3 we introduce a fractional differential ring, in the Section 4 we give some result of GFD, in the Section 5 we give a definition of partial fractional differential derivative, in the Section 6 we give a definition of GFD when $\alpha \in (n, n + 1]$.

2. Fractional derivative

Let $\alpha \in (0, 1]$ be a fractional number, we want to give a definition of generalized fractional derivative of order α for a differentiable function f. We denote $\alpha-$th derivative of f by $D^\alpha(f)$ and we denote the first derivative of f by $D(f)$.

We begin the present section listing previous definition of fractional derivative; later we present our proposal of generalized one showing how it contains the once already described. We finish the section providing some examples.

1. The Caputo fractional derivative was defined by Michele Caputo in [24]:

$$D^\alpha(f) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(x)}{(t-x)^\alpha} dx. \tag{1}$$

2. The conformable fractional derivative was defined by Khalil, Al Horani, Yousef and Sababheh in [11]:

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + t^{1-\alpha}\varepsilon) - f(t)}{\varepsilon}. \tag{2}$$

3. The conformable fractional derivative was defined by Anderson and Ulness in [25]:

$$D^\alpha f(t) = (1 - \alpha) \mid t \mid^\alpha f(t) + \alpha \mid t \mid^{1-\alpha} Df. \tag{3}$$

4. The fractional derivative was defined by Udita N.Katugampola in [17]:

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(te^{\varepsilon t^{-\alpha}}) - f(t)}{\varepsilon}. \tag{4}$$

5. The fractional derivative was defined by Guebbai and Ghiat in [20] for an increasing and positive function f:

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \left(\frac{f(t + f(t)^{\frac{1-\alpha}{\alpha}}\varepsilon) - f(t)}{\varepsilon} \right)^\alpha. \tag{5}$$

6. The conformable ratio derivative was defined by Camrud in [14] for a function $f(t) \geq 0$ with $Df(t) \geq 0$:

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} f(t)^{1-\alpha} \left(\frac{f(t + \varepsilon) - f(t)}{\varepsilon} \right)^\alpha. \tag{6}$$

From all these definitions, we propose a definition that unifies almost all of them.

Definition 1. Given a differentiable function $f : [0, \infty) \to \mathbb{R}$, **the generalized fractional derivative**(GFD) for $\alpha \in (0, 1]$ at point t is defined by:

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + w_{t,\alpha} t^{1-\alpha}\varepsilon) - f(t)}{\varepsilon},$$

where $w_{t,\alpha}$ is a function that may depend on α and t.

Remark 1. As a consequence of Definition 1 we can see

$$D^\alpha f(t) = w_{t,\alpha} t^{1-\alpha} Df(t).$$

Definition 2. A differentiable function $f : [0, \infty) \to \mathbb{R}$ is said to be $\alpha-$**generalized fractional differentiable function** over $[0, \infty)$ if it exists $D^\alpha(f)(t)$ for all $t \in [0, \infty)$ for $\alpha \in (0, 1]$.

We denote $C^\alpha[0, \infty)$ the set of $\alpha-$generalized differentiable functions with real values in the interval $[0, \infty)$ in variable t. The set $(C^\alpha[0, \infty), +, .)$ is a ring. In the following we want to see the relation between GFD and the others definitions:

1. The fractional derivative of Khalil, Al Horani, Yousef and Sababheh in [11] is a particular case of GFD where $w_{t,\alpha} = 1$.
2. The fractional derivative of Anderson and Ulness in [25] is a particular case of GFD where

$$w_{t,\alpha} = \frac{(1 - \alpha)t^\alpha f(t) + \alpha t^{1-\alpha} Df}{\alpha t^{1-\alpha}}.$$

In this fractional derivative $w_{t,\alpha}$ depends on α and t.
3. The fractional derivatives of Guebbai and Ghiat in [20] and Camrud in [14] are particular cases of GFD where $w_{t,\alpha} = \left(\frac{tDf}{f} \right)^{\alpha-1}$.

We are particularly interested in discussing GFD where $w_{t,\alpha} = g(t, \alpha)\tau^{\alpha-1}$ such that $g : [0, \infty) \times (0, 1] \to \mathbb{R}$ is a function and τ is the characteristic of system with the properties

$$w_{t,\alpha} = 1 \quad \text{if and only if} \quad \alpha = 1.$$

If the system is periodic with period T, then we have $\tau = T$. In the quantum systems τ is the Bohr radius and in astronomy τ is the light year. The characteristic of system τ depends on the systems and the derivative. If t is time, τ is time too. If t is space, τ is space too. In fact the unit of t is τ, i.e., $t = c\tau$ where c is a constant. In the general $\tau = 1$.

Example 1. Let $\alpha, \beta \in (0,1]$. Let f, h be two functions in $C^\alpha[0, \infty)$. We suppose $w_{t,\alpha} = g(t,\alpha)\tau^{\alpha-1}$ with $\tau = 1$.

1. If $g(t,\alpha)$ is a function with $g(0,0) = 0$, then $\lim_{\alpha \to 0} D^\alpha(f) = 0$.
2. If $g(t,\alpha) = \alpha$, we have the chain rule

$$D^\alpha(f \circ h) = \frac{t^{\alpha-1}}{\alpha} D^\alpha\left(f(h)\right) D^\alpha(h).$$

Example 2. We want to present the corresponding figure to the generalized fractional derivatives for $\alpha = \frac{3}{4}$ for a trigonometric, using all the fractional derivative definitions that we have already mentioned in this article. It can be seen from all the figures that in principle these definitions do not find a reason to discard them. That is, they have a fairly reasonable behavior. We consider $f(t) = \sin(2t)$, the graph of f can be seen in Figure 1.
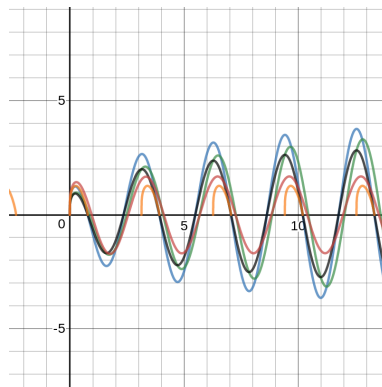

Figure 1. red:Caputo, green:Khalil *et al.*, blue:Anderson *et al.*, orange:Guebbai *et al.*, black:GFD when $w_{t,\alpha} = \alpha$.

3. Generalized fractional differential ring

In this section we want to stress out that instead of defining a new derivative, we focus on the notion of differentiable operator and the ring that it carries with.

Definition 3. Let R be a commutative ring with unity. A derivation on R is a map $d : R \to R$ that satisfies $d(a + b) = d(a) + d(b)$ and $d(ab) = d(a)b + ad(b), \forall a, b \in R$. The pair (R, d) is called a differential ring (see [26]).

Theorem 1. *Let $\alpha \in (0,1]$, then the ring $C^\alpha[0, \infty)$ with operator $D^\alpha : C^\alpha[0, \infty) \to C^\alpha[0, \infty)$ is a differential ring.*

Proof. Since $C^\alpha[0, \infty)$ is a commutative ring with unity $f(t) = 1$ and the derivation D^α for $\alpha \in (0,1]$ satisfies following properties from Remark 1

1. $D^\alpha(af_1 + bf_2) = aD^\alpha(f_1) + bD^\alpha(f_2), \forall f_1, f_2 \in C^\alpha[0, \infty), \forall a, b \in \mathbb{R}$,
2. $D^\alpha(f_1 f_2) = f_1 D^\alpha(f_2) + f_2 D^\alpha(f_1), \forall f_1, f_2 \in C^\alpha[0, \infty)$.
□

Let $\alpha \in (0,1]$ be a fractional number, and $f_1, f_2 \in C^\alpha[0, \infty)$ be two functions, then GFD has the following properties:

1. $D^\alpha\left(\frac{f_1}{f_2}\right) = \frac{f_2 D^\alpha f_1 - f_1 D^\alpha f_2}{f_2^2}$.
2. $D^\alpha(f_1 \circ f_2) = \frac{t^{\alpha-1}}{w_{t,\alpha}} D^\alpha\left(f_1(f_2)\right) D^\alpha(f_2)$.
3. $D^{\alpha+\beta}(f_1) = \frac{w_{t,\alpha} w_{t,\beta} t}{w_{t,\alpha+\beta}} D^\alpha D^\beta(f_1)$.

It is easy to see these properties from Remark 1 that if $w_{t,\alpha} = t^{1-\alpha}$ we have the equality

$$D^\alpha (f_1 \circ f_2) = D^\alpha (f_1(f_2)) D^\alpha (f_2).$$

If $\frac{w_{t,\alpha+\beta}}{w_{t,\alpha} w_{t,\beta}} = t$ we have the equality

$$D^{\alpha+\beta}(f_1) = D^\alpha D^\beta(f_1), \forall \alpha, \beta \in (0,1].$$

Parts 4 and 5 of the properties imply that we can create function spaces with different algebras using different expressions for $w_{t,\alpha}$.

By considering previous properties of GFD we called $C^\alpha[0, \infty)$ a $w_{t,\alpha}-$**generalized fractional differential ring** of functions and we denote it by $(C^\alpha[0, \infty), D^\alpha, w_{t,\alpha})$. Let $I \subset C^\alpha[0, \infty)$ be an ideal. If $D^\alpha(I) \subset I$ then the ideal I is called a $w_{t,\alpha}-$**generalized fractional differential ideal**. By using the previous properties we can see the following result:

Theorem 2. *Let $\alpha \in (0,1]$, then, associated to any α and any $w_{t,\alpha}$ there exists a fractional differential ring.*

4. Some results of Generalized Fractional Derivative

Let $\alpha \in (0,1]$ be a fractional number and $t \in [0, \infty)$ then GFD has the following properties:

1. $D^\alpha(\frac{t^\alpha}{\alpha w_{t,\alpha}}) = 1$,
2. $D^\alpha(\sin(\frac{t^\alpha}{\alpha w_{t,\alpha}})) = \cos(\frac{t^\alpha}{\alpha w_{t,\alpha}})$,
3. $D^\alpha(\cos(\frac{t^\alpha}{\alpha w_{t,\alpha}})) = -\sin(\frac{t^\alpha}{\alpha w_{t,\alpha}})$,
4. $D^\alpha(e^{(\frac{t^\alpha}{\alpha w_{t,\alpha}})}) = e^{(\frac{t^\alpha}{\alpha w_{t,\alpha}})}$.

Theorem 3. *(Rolle's Theorem for $\alpha-$Generalized Fractional Differentiable Functions)*
Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function with the properties that

1. *f is continuous on $[a,b]$,*
2. *f is $\alpha-$generalized fractional differentiable on (a,b) for some $\alpha \in (0,1]$,*
3. *$f(a) = f(b)$.*

Then, there exist $c \in (a,b)$ such that $D^\alpha f(c) = 0$.

Proof. Since f is continuous on $[a,b]$ and $f(a) = f(b)$, then the function f has a local extreme in a point $c \in (a,b)$ and

$$D^\alpha f(c) = \lim_{\varepsilon \to 0^+} \frac{f(c + w_{t,\alpha} c^{1-\alpha}\varepsilon) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f(c + w_{t,\alpha} c^{1-\alpha}\varepsilon) - f(c)}{\varepsilon}.$$

But two limits have different signs, so $D^\alpha f(c) = 0$. $\square$

Theorem 4. *(Mean Value Theorem for $\alpha-$Generalized Fractional Differentiable Functions) Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function with the properties that*

1. *f is continuous on $[a,b]$,*
2. *f is $\alpha-$Generalized fractional differentiable on (a,b) for some $\alpha \in (0,1]$.*

Then, there exists $c \in (a,b)$ such that $D^\alpha f(c) = \frac{\alpha w_{t,\alpha}(f(b)-f(a))}{b-a}$.

Proof. Consider function

$$h(t) = f(t) - f(a) - \frac{\alpha w_{t,\alpha}(f(b) - f(a))}{b - a} \left(\frac{t^\alpha}{\alpha w_{t,\alpha}} - \frac{a^\alpha}{\alpha w_{t,\alpha}} \right).$$

Then, the function h satisfies the conditions of the fractional Rolle's Theorem. Hence, there exists $c \in (a,b)$ such that $D^\alpha h(c) = 0$. We have the result since

$$D^\alpha h(c) = D^\alpha f(c) - \frac{\alpha w_{t,\alpha}(f(b) - f(a))}{b - a}(1) = 0.$$

$\square$

5. Generalized Partial Fractional Derivative

In this section we introduce a partial fractional derivative of first and second order. Also we introduce a partial fractional differential ring.

Definition 4. Let $f(t_1, \cdots, t_n) : [0, \infty)^n \to \mathbb{R}$ be a function with n variables such that $\forall i$, then there exists the partial derivative of f respect to t_i. Let $\alpha \in (0, 1]$ be a fractional number. We define $\alpha-$**generalized partial fractional derivative**(GPFD) of f with respect to t_i at point $t = (t_1, \ldots, t_n)$

$$\frac{\partial^\alpha f(t)}{\partial t_i^\alpha} = \lim_{\varepsilon \to 0} \frac{f(t_1, \ldots, t_i + w_{t_i,\alpha} t_i^{1-\alpha} \varepsilon, \ldots, t_n) - f(t)}{\varepsilon},$$

where $w_{t_i,\alpha}$ can be a function depend on α and t_i.

Remark 2. As a consequence of Definition 2 we can see for $\alpha \in (0, 1]$ and $1 \leq i \leq n$:

$$\frac{\partial^\alpha f}{\partial t_i^\alpha}(t) = w_{t_i,\alpha}(t_i)^{1-\alpha} \frac{\partial f}{\partial t_i}(t).$$

Let $\alpha \in (0, 1]$ and $1 \leq i \leq n$. A partial differentiable function $f : [0, \infty)^n \to \mathbb{R}$ is said to be a $\alpha-$**generalized partial fractional differentiable function** respect to t_i over $[0, \infty)$ if there exists $\frac{\partial^\alpha f(t)}{\partial t_i^\alpha}$ for all $t \in [0, \infty)$. We denote by $C_i^\alpha[0, \infty)^n$ the set of $\alpha-$generalized partial fractional differentiable functions respect to t_i with real values in the interval $[0, \infty)^n$ in variable $t = (t_1, \ldots, t_n)$. The set $(C_i^\alpha[0, \infty)^n, +, .)$ is a ring.

Theorem 5. *Let $\alpha \in (0, 1]$ and $1 \leq i \leq n$. The ring $C_i^\alpha[0, \infty)^n$ with operator*

$$\frac{\partial^\alpha}{\partial t_i^\alpha} : C_i^\alpha[0, \infty)^n \to C_i^\alpha[0, \infty)^n$$

is a differential ring.

Proof. Since the ring $C_i^\alpha[0, \infty)^n$ is a commutative ring with unity $f(t_1, \ldots, t_n) = 1$ and the derivation $\frac{\partial^\alpha}{\partial t_i^\alpha}$ for $\alpha \in (0, 1]$ satisfies the following properties from Remark 2;

1. $\frac{\partial^\alpha (f_1 + f_2)}{\partial t_i^\alpha} = \frac{\partial^\alpha (f_1)}{\partial t_i^\alpha} + \frac{\partial^\alpha (f_2)}{\partial t_i^\alpha} \quad f_1, f_2 \in C^\alpha[0, \infty)^n$,
2. $\frac{\partial^\alpha (f_1 f_2)}{\partial t_i^\alpha} = f_1 \frac{\partial^\alpha f_2}{\partial t_i^\alpha} + f_2 \frac{\partial^\alpha f_1}{\partial t_i^\alpha} \quad f_1, f_2 \in C^\alpha[0, \infty)^n$.
□

Let $\alpha \in (0, 1]$ and $1 \leq i \leq n$. Further let $f_1, f_2 \in C^\alpha[0, \infty)^n$ be two functions, then GPFD has the following properties from Remark 2:

1. $\frac{\partial^\alpha (\frac{f_1}{f_2})}{\partial t_i^\alpha} = \frac{f_2 \frac{\partial^\alpha f_1}{\partial t_i^\alpha} - f_1 \frac{\partial^\alpha f_2}{\partial t_i^\alpha}}{f_2^2}$,
2. $\frac{\partial^\alpha f_1 \circ f_2}{\partial t_i^\alpha} = \frac{t_i^{\alpha-1}}{w_{t_i,\alpha}} \frac{\partial^\alpha (f_1(f_2))}{\partial t_i^\alpha} \frac{\partial^\alpha (f_2)}{\partial t_i^\alpha}$,
3. $\frac{\partial^{\alpha+\beta}(f_1)}{\partial t_i^\alpha} = \frac{w_{t_i,\alpha} w_{t_i,\beta} t_i}{w_{t_i,\alpha+\beta}} \frac{\partial^\alpha}{\partial t_i^\alpha} \frac{\partial^\alpha (f_1)}{\partial t_i^\alpha}$.

By considering previous properties of GPFD we called the ring $C_i^\alpha[0, \infty)^n$ a $w_{t_i,\alpha}-$ **generalized partial fractional differential ring**. We denote it by $(C_i^\alpha[0, \infty)^n, \frac{\partial^\alpha}{\partial t_i^\alpha}, w_{t_i,\alpha})$.

We can see the following result by using the previous properties:

Theorem 6. *Let $\alpha \in (0, 1]$ and $1 \leq i \leq n$. Associated to any α and any $w_{t_i,\alpha}$ there is a partial fractional differential ring.*

Example 3. Let $f(t_1, t_2) = t_1^3 \sin(t_2)$ and $\alpha \in [0, 1)$ then we have

$$\frac{\partial^\alpha f}{\partial t_1^\alpha} = w_{t,\alpha}(t_1)^{1-\alpha}(3t_1^2)\sin(t_2).$$

Definition 5. Let $\alpha \in (0,1]$ be a fractional number. We define $\alpha-$**generalized partial fractional derivative of second order** with respect to t_i and t_j at point $t = (t_1, \cdots, t_n)$ is

$$\frac{\partial^{\alpha^2} f(t)}{\partial t_j^\alpha \partial t_i^\alpha} = \frac{\partial^\alpha}{\partial t_j^\alpha}\left(\frac{\partial^\alpha f(t)}{\partial t_i^\alpha}\right) = \lim_{\varepsilon \to 0} \frac{\frac{\partial^\alpha f}{\partial t_i^\alpha}(t_1,\ldots,t_j+w_{t_j,\alpha}t_j^{1-\alpha}\varepsilon,\ldots,t_n) - \frac{\partial^\alpha f(t)}{\partial t_i^\alpha}}{\varepsilon}.$$

Remark 3. As a consequence of Definition 5 we can see for $\alpha \in (0,1]$ and $1 \le i,j \le n$;

$$\frac{\partial^{\alpha^2} f(t)}{\partial t_j^\alpha \partial t_i^\alpha} = w_{t_j,\alpha} w_{t_i,\alpha}(t_j t_i)^{1-\alpha} \frac{\partial}{\partial t_j}\left(\frac{\partial f}{\partial t_i}(t)\right).$$

A partial differentiable function of second order $f : [0,\infty)^n \to \mathbb{R}$ is said to be a $\alpha-$generalized fractional partial differentiable function of second order respect to t_i and t_j over $[0,\infty)$ if there exists $\frac{\partial^{\alpha^2} f(t)}{\partial t_j^\alpha \partial t_i^\alpha}$ for all $t \in [0,\infty)$. We denote $C_{i,j}^{\alpha^2}[0,\infty)^n$ the set of $\alpha-$generalized partial fractional differentiable functions of second order respect to t_i and t_j with real values in the interval $[0,\infty)^n$ in variable $t = (t_1,\ldots,t_n)$. The set $(C_{i,j}^{\alpha^2}[0,\infty)^n,+,.)$ is a ring.

6. Generalized Fractional Derivative for $\alpha \in (n, n+1]$

In this section we define a fraction differential derivative for $\alpha \in (n, n+1]$.

Definition 6. Let $\alpha \in (n, n+1]$ be a fractional number for $n \in \mathbb{N}$ and $f : [0.\infty) \to \mathbb{R}$ be a $n-$ differentiable. The **generalized fractional derivative** of order α is defined by

$$D^\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f^{[\alpha]-1}(t+w_{t,\alpha}t^{[\alpha]-\alpha}\varepsilon) - f^{[\alpha]-1}(t)}{\varepsilon},$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

As a consequence of Definition 6 we can see

$$D^\alpha(f) = w_{t,\alpha} t^{[\alpha]-\alpha} D^{[\alpha]}(f),$$

where $\alpha \in (n, n+1]$.

Let $n < \alpha \le n+1$. A function $f : [0,\infty) \to \mathbb{R}$ is said to be $\alpha-$generalized differentiable over $[0,\infty)$ if there exists $D^\alpha(f)(t)$ for all $t \in [0,\infty)$. We denote $C^\alpha[0,\infty)$ the set of $\alpha-$generalized fractional differentiable functions with real values in the interval $[0,\infty)$ in variable t. The set $(C^\alpha[0,\infty),+,.)$ is a ring.

Theorem 7. *The ring* $(C^\alpha[0,\infty),+,.)$ *with operator* D^α *is not a differential ring for fractional number* $\alpha \in (n, n+1]$.

Proof. Since $D^\alpha(fg) \ne fD^\alpha g + gD^\alpha(f)$ for every $f,g \in C^\alpha[0,\infty)$. $\square$

7. α-Fractional Taylor Series

There are some articles about fractional Taylor series see ([12,27–29]). In this section we use GFD to define a fractional taylor series for a function $f \in C^r[0,\infty)$ for every fractional number r.

Let $0 < \alpha < 1$, we define the α-fractional taylor series of f at real number x_0

$$f(x) = f(x_0) + \sum_{i=1}^\infty \frac{D^i f(x_0)}{w_{x,\alpha}\overline{(\alpha+i-1)!}}(x-x_0)^{\alpha+i-1},$$

where $\overline{(\alpha+i-1)!} = \alpha(\alpha+1)\cdots(\alpha+i-1)$.

Let $1 < \alpha \le 2$, we define the α-fractional taylor series of f at real number x_0

$$f(x) = f(x_0) + Df(x_0)(x-x_0) + \sum_{i=2}^\infty \frac{D^i f(x_0)(x-x_0)^{\alpha+i-2}}{w_{x,\alpha}\overline{(\alpha+i-2)!}},$$

where $\overline{(\alpha+i-2)!} = (\alpha-1)\alpha(\alpha+1)\cdots(\alpha+i-2)$.

Let $n < \alpha \le n+1$ such that $\alpha = n + A$ with $0 < A < 1$. We define the α-fractional taylor series of f at real number x_0,

$$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{D^i f(x_0)(x-x_0)^i}{i!} + \sum_{i=n+1}^{\infty} \frac{D^i f(x_0)}{w_{x,\alpha}\overline{(A+i-1)!}}(x-x_0)^{A+i-1},$$

where $\alpha = n + A$, $\overline{(A+i-1)!} = A(A+1)\cdots(A+i-1)$.

8. Application to differential equations

There are some articles for applications of fractional differential derivative such as [3,6,7,9]. In this section we solve some (partial) fractional differential equations by using our definitions. Firstly we solve the fractional differential equations with the form

$$aD^\alpha y + by = c, \tag{7}$$

where $y = f(t)$ be a differentiable function and $0 < \alpha < 1$.

By substituting GFD in the Equation (7) we have

$$aw_{t,\alpha}t^{1-\alpha}Dy + by = c \Longrightarrow Dy + \frac{bt^{\alpha-1}}{aw_{t,\alpha}}y = \frac{t^{\alpha-1}c}{aw_{t,\alpha}},$$

the solutions of this equation have the form $y(t) = \frac{c}{b} + c_1 e^{\left(-\frac{bt^\alpha}{aw_{t,\alpha}\alpha}\right)}$.

Example 4. We consider the partial fractional differential equation with boundary conditions

$$\begin{cases} u_t + 2\sqrt[3]{xu_x} + u = x^2, t > 0 \\ u(t,0) = 0, \\ u(0,x) = 0, \end{cases} \tag{8}$$

where $u(x,t)$ be a differentiable function respect to x and t, $u(x,t)$ be a $\frac{1}{3}-$partial fractional differentiable function of first order respect to x, $u_t = \frac{\partial u}{\partial t}$ and $\sqrt[3]{u_x} = \frac{\partial^{\frac{1}{3}} u}{\partial x^{\frac{1}{3}}}$. For $w_{t,\frac{1}{3}} = \frac{1}{3}$, by using Remark 2 we can write

$$u_t + 2\sqrt[3]{xu_x} + u = x^2 \Longrightarrow u_t + 2\sqrt[3]{x}w_{t,\alpha}x^{1-\alpha}u_x + u = x^2 \Longrightarrow u_t + \frac{2}{3}xu_x + u = x^2.$$

We solve this equation by taking Laplace transform of equation with respect to t. We denote by $U(x,s)$ the Laplace of $u(x,t)$ with respect to t, we have the following equation

$$\begin{cases} sU(x,s) - U(x,0) + \frac{2}{3}xU_x(x,s) + U(x,s) = x^2 s \\ U(t,0) = 0, \\ U(0,x) = 0. \end{cases} \tag{9}$$

Then

$$U_x + \frac{3+3s}{2x}U = \frac{3x}{2s} \Longrightarrow U(x,s) = \frac{3x^2}{s(3s+7)} + c(s)x^{\frac{-3-3s}{2}}.$$

By substituting $U(0,x) = 0$, we have $c(s) = 0$, then $U(x,s) = x^2(\frac{3}{7s} + \frac{9}{7(3s+7)})$. The solution of equation is $u(x,t) = \frac{x^2}{7}(1 - e^{\frac{-7}{3}})$.

Example 5. We consider the partial fractional differential equation of second order;

$$\sqrt[5]{u_{xt}} + 2\frac{u}{x} = 0, \tag{10}$$

where $u(x,t)$ be a $\frac{1}{5}-$fractional partial differentiable function of second order respect to t,x. For $w_{x,\frac{1}{5}} = x^2$, $w_{t,\frac{1}{5}} = \frac{1}{\sqrt[3]{t}}$ by using remark 3, we have

$$x^2\sqrt[5]{x^4}\sqrt[5]{t^4}u_{xt} + 2\sqrt[3]{t}u = 0. \tag{11}$$

We consider a solution of this differential equation of the form $u(x,t) = f(x)g(t)$ such that f a function depends on x and g a function depends on t. By substituting $u(x,t)$ in the Equation (11) we have

$$x^2 \sqrt[5]{x^4} \sqrt[5]{t^4} Df.Dg + 3\sqrt[3]{t} fg = 0. \tag{12}$$

We can write Equation (12) in a form that divide the functions of t and x:

$$\frac{x^2 \sqrt[5]{x^4} Df}{f} = -\frac{2\sqrt[3]{t}g}{\sqrt[5]{t^4}Dg}, \tag{13}$$

two sides of the Equality (13) is a constant k. We have

$$\begin{cases} \frac{Df}{f} = \frac{k}{\sqrt[5]{x^{14}}} \rightarrow Lnf = k\sqrt[15]{x} + c_1 \rightarrow f = \exp(k\sqrt[15]{x} + c_1), \\ \frac{Dg}{g} = \frac{-2}{k\sqrt[15]{t^7}} \rightarrow Lng = \frac{-2\sqrt[15]{t^8}}{k} + c_2 \rightarrow g = \exp(\frac{-2\sqrt[15]{t^8}}{k} + c_2). \end{cases} \tag{14}$$

The solution of Equation (10) has the form

$$u(x,t) = \exp(k\sqrt[15]{x} + c_1)\exp(\frac{-2\sqrt[15]{t^8}}{k} + c_2) = c\exp(k\sqrt[15]{x} + \frac{-2\sqrt[15]{t^8}}{k}).$$

9. Conclusion

We defined a generalized fractional derivative (GFD). We showed that the previous derivatives are particular cases. We also showed how it is possible to have infinite fractional derivatives with their algebra. We present the fractional differential ring, the fractional partial derivatives and their applications.

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