Open Journal of Mathematical Sciences

PSR Press

Article

Generalized fractional Hadamard type inequalities for Q_s–class functions of the second kind

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Academic Editor: MERVE Ilkhan Kara
Received: 26 March 2021; Accepted: 9 June 2021; Published: 16 July 2021.

Abstract: New Hadamard type inequalities for a class of s-Godunova–Levin functions of the second kind for fractional integrals are obtained. These new estimates extend and generalize some existing results for the Q–class and P–class functions. The generalized case for the Katugampola fractional integrals are also given.

Keywords: Q_s–class functions; Hadamard–type inequalities; Generalized Katugampola fractional integrals; Generalized Riemann–Liouville fractional integral.

MSC: 35A23; 26E70; 34N05.

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1. Introduction and Preliminaries

Hermite–Hadamard (HH) inequalities and other related inequalities for convex functions have been extensively studied by different researchers, see [1,2] and their references. A much broader class of functions known as Q–class functions was proposed by Godunova and Levin [3,4]. This class of functions is very important because it contains all nonnegative monotone and nonnegative convex functions. Motivated by the fact that this class of functions is much bigger and broader than the class of convex functions (which many authors have given different HH and HH type inequalities on); we, therefore, present, extend and generalize the HH inequalities on this broader class of functions for fractional integrals.

Definition 1 ([5–9])**.** A nonnegative function $f : I \to \mathbb{R}$ is said to be a Q–class function, if for all $x, y \in I$, and $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}. \tag{1}$$

Definition 2 ([5,6])**.** Let D be a subset of $\mathbb{R}$ with at least two elements. A function $f : I \to \mathbb{R}$ is said to be a Shur function if

$$f(x)(x-y)(x-z) + f(y)(y-x)(y-z) + f(z)(z-x)(z-y) \ge 0, \tag{2}$$

for all $x, y, z \in D$.

Remark 1. Godunova and Levin [3] also showed that the class of Schur functions and the Q–class functions are equivalent. That is, (1) and (2) concide.

Definition 3 ([5,6])**.** A nonnegative function $f : I \to \mathbb{R}$ is said to be a P–function, if $\forall\, x, y \in I$, $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y). \tag{3}$$

It is also known that $P(I) \subset Q(I)$, and contains all nonnegative monotone, convex and quasi-convex functions: nonnegative functions satisfying

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}.$$

The following Hamadard type inequalities have already been proved:

Theorem 1 ([4–6])**.** *Let $f \in Q(I)$, $a, b \in I$, with $a < b$ and $f \in L^1[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(x)dx,$$

and

$$\frac{1}{b-a}\int_a^b p(x)f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where $p(x) = \frac{(b-x)(x-a)}{(b-a)^2}$.

Next, we state some generalizations of the Q–class function known as the s-Godunova–Levin functions (Q_s–class functions):

Definition 4 ([7–9])**.** A nonnegative function $f : I \to \mathbb{R}$ is said to be a Q_s–class functions of first kind if for $s \in (0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda^s} + \frac{f(y)}{1-\lambda^s}, \tag{4}$$

$\forall\, x,\, y \in I,\, \lambda \in (0,1)$.

Remark 2. Taking $s = 1$ in (4), we obtain the definition of Q–class function in (1).

Definition 5. A function $f : I \to \mathbb{R}$ is said to be a Q_s–class functions of second kind if for $s \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda^s} + \frac{f(y)}{(1-\lambda)^s}, \tag{5}$$

$\forall x,\, y \in I,\, \lambda \in (0,1)$.

Remark 3. Observe that $s = 0$ in (5) gives the definition of P–class function in (3), and $s = 1$ gives the definition of Q–class function in (1).

The paper is organized as follows. Section 2 contains the main results of the paper. In Section 3, we give a concise summary of the paper.

2. Main Results

Our aim is to extend and generalize the result of Theorem 1 to s–Godunova–Levin functions of second kind given in (5):

2.1. Some Auxilliary Results

Definition 6. A function $f(t)$ is said to be in $L_{p,r}[a,b]$ if

$$\left(\int_a^b |f(t)|^p t^r dt\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,\; r \geq 0,$$

where $L_{1,0}[a,b] = L_1[a,b]$.

Theorem 2. *Let $f \in Q_s(I)$, $a,\, b \in I$, with $a < b$ and $f \in L_1[a,b]$ satisfying (5). Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{s+1}}{b-a}\int_a^b f(x)dx,$$

and

$$\frac{1}{b-a}\int_a^b p_s(x)f(x)dx \leq \frac{f(a)+f(b)}{1+s}, \tag{6}$$

where $p_s(x) = \frac{(b-x)^s(x-a)^s}{(b-a)^{2s}}$.

Proof. Let $\lambda = \frac{1}{2}$ in (5) to get $2^s\big(f(x)+f(y)\big) \geq f\big(\frac{x+y}{2}\big)$. Define $x = ta + (1-t)b,\ y = (1-t)a + tb$, then

$$2^s\big[f(ta+(1-t)b) + f((1-t)a+tb)\big] \geq f\left(\frac{a+b}{2}\right). \tag{7}$$

Integrate both sides of (7) over $t \in [0,1]$ to obtain

$$2^s\left[\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt\right] \geq f\left(\frac{a+b}{2}\right). \tag{8}$$

Let $I_1 = \displaystyle\int_0^1 f(ta+(1-t)b)dt$ and $I_2 = \displaystyle\int_0^1 f((1-t)a+tb)dt$. For I_1, let $z = ta + (1-t)b,\ z - b = (a-b)t$ and $dz = (a-b)dt$. Also, when $t = 0$, $z = b$ and when $t = 1$, $z = a$. Thus,

$$I_1 = \int_b^a f(z)\frac{dz}{a-b} = \frac{1}{b-a}\int_a^b f(z)dz.$$

Similarly, for I_2, we let $u = (1-t)a + tb),\ u - a = (b-a)t$ and $du = (b-a)dt$; when $t = 0$, $u = a$ and when $t = 1$, $u = b$. Therefore,

$$I_2 = \int_a^b f(u)\frac{du}{b-a} = \frac{1}{b-a}\int_a^b f(u)du.$$

Combining I_1 and I_2, Inequality (8) becomes $2^s.2\displaystyle\int_a^b f(x)dx \geq f\left(\frac{a+b}{2}\right)$, and the first part of the result follows. For the second part of the proof, we multiply (5) by $\lambda^s(1-\lambda)^s$,

$$\lambda^s(1-\lambda)^s f(\lambda a + (1-\lambda)b) \leq (1-\lambda)^s f(a) + \lambda^s f(b). \tag{9}$$

Similarly,

$$\lambda^s(1-\lambda)^s f((1-\lambda)a + \lambda b) \leq \lambda^s f(a) + (1-\lambda)^s f(b). \tag{10}$$

Now, add (9) and (10), and integrate over $\lambda \in [0,1]$:

$$\int_0^1 \lambda^s(1-\lambda)^s\Big[f(\lambda a + (1-\lambda)b) + f((1-\lambda)a + \lambda b)\Big]d\lambda \leq [f(a)+f(b)]\int_0^1 [\lambda^s + (1-\lambda)^s]d\lambda.$$

Evaluating the integrals as before, we have

$$\frac{2}{b-a}\int_a^b \frac{(b-x)^s(x-a)^s}{(b-a)^{2s}} f(x)dx \leq \frac{2\big(f(a)+f(b)\big)}{1+s}.$$

□

One can apply the Integral Chebyshev inequality, to get alternative inequalities of (6) of Theorem 2:

Corollary 1. *Suppose that the hypotheses of Theorem 2 hold. Then*

$$\frac{\Gamma^2(1+s)}{\Gamma(2+2s)}\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{1+s},$$

and

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{1-s},\ s \in [0,1).$$

Proof. Recall from the proof of Theorem 2, that

$$\int_0^1 \lambda^s(1-\lambda)^s\Big[f(\lambda a + (1-\lambda)b) + f((1-\lambda)a + \lambda b)\Big]d\lambda \leq [f(a)+f(b)]\int_0^1 [\lambda^s + (1-\lambda)^s]d\lambda.$$

Now, applying the Integral Chebyshev inequality on the integrals

$$\int_0^1 \lambda^s(1-\lambda)^s f(\lambda a + (1-\lambda)b)d\lambda,$$

and
$$\int_0^1 \lambda^s(1-\lambda)^s f((1-\lambda)a+\lambda b)d\lambda.$$
Thus,
$$\begin{aligned}\int_0^1 \lambda^s(1-\lambda)^s f(\lambda a+(1-\lambda)b)d\lambda &\geq \int_0^1 \lambda^s(1-\lambda)^s d\lambda \int_0^1 f(\lambda a+(1-\lambda)b)d\lambda\\ &= \frac{\Gamma^2(1+s)}{\Gamma(2+2s)}\frac{1}{b-a}\int_a^b f(x)dx.\end{aligned}$$
Similarly,
$$\int_0^1 \lambda^s(1-\lambda)^s f((1-\lambda)a+\lambda b)d\lambda \geq \frac{\Gamma^2(1+s)}{\Gamma(2+2s)}\frac{1}{b-a}\int_a^b f(x)dx.$$
So, we obtain that
$$\frac{\Gamma^2(1+s)}{\Gamma(2+2s)}\frac{2}{b-a}\int_a^b f(x)dx \leq \frac{2\big(f(a)+f(b)\big)}{1+s},$$
and the first inequality follows.

Next, we write Inequality (5) for a, b as follows:
$$f(\lambda a+(1-\lambda)b) \leq \lambda^{-s}f(a)+(1-\lambda)^{-s}f(b),$$
and integrate over $\lambda \in [0,1]$ to get
$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx &= \int_0^1 f(\lambda a+(1-\lambda)b)d\lambda\\ &\leq f(a)\int_0^1 \lambda^{-s}d\lambda + f(b)\int_0^1 (1-\lambda)^{-s}d\lambda\\ &= \frac{f(a)+f(b)}{1-s}.\end{aligned}$$
□

2.2. Fractional Hadamard type inequalities

Next, we extend the results for fractional integrals:

Definition 7 ([10,11])**.** If $f \in L_1([a,b])$. Then the right (and respectively the left) Riemann–Liouville fractional integral of order $\alpha \geq 0$ is given by
$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \;\; t \in [a,b],$$
and
$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds, \;\; t \in [a,b].$$

Theorem 3. *Let $f \in Q_s(I)$, $a, b \in I$, with $a < b$ and $f \in L_1[a,b]$ satisfying (5). Then*
$$f\left(\frac{a+b}{2}\right) \leq \frac{2^s\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{b^-}f(a)+I^{\alpha}_{a^+}f(b)\right],$$
and
$$\frac{\Gamma^2(s\alpha+1)}{(b-a)^{s(\alpha+1)+1}\left[\Gamma(1+s\alpha)+\Gamma(1+s)\Gamma(1-s(1-\alpha))\right]}\left[I^{s\alpha+1}_{b^-}\left[(b-a)^s f(a)+I^{s\alpha+1}_{a^+}\left[(b-a)^s f(b)\right]\right.\right] \leq \frac{f(a)+f(b)}{1+s\alpha}.$$

Proof. Following similar steps as above, for $\lambda = \frac{1}{2}$, $x = ta+(1-t)b$, $y = (1-t)a+tb$, we have
$$2^s\left[f(ta+(1-t)b)+f((1-t)a+tb)\right] \geq f\left(\frac{a+b}{2}\right).$$

Multiply through by $t^{\alpha-1}$ and integrate over $t \in [0,1]$ to obtain

$$2^s\left[\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt\right] \geq \frac{1}{\alpha} f\left(\frac{a+b}{2}\right). \tag{11}$$

Evaluating the integrals:

$$\begin{aligned}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt &= \int_b^a \left(\frac{z-b}{a-b}\right)^{\alpha-1} f(z)\frac{dz}{a-b}\\ &= \frac{1}{(b-a)^\alpha}\int_a^b (b-z)^{\alpha-1}dz\\ &= \frac{\Gamma(\alpha)}{(b-a)^\alpha} I_{b-}^\alpha f(a).\end{aligned}$$

Similarly, $\int_0^1 t^{\alpha-1} f((1-t)a+tb)dt = \frac{\Gamma(\alpha)}{(b-a)^\alpha} I_{a+}^\alpha f(b)$. Thus, inequality (11) becomes

$$2^s \frac{\Gamma(\alpha)}{(b-a)^\alpha}\left[I_{b-}^\alpha f(a) + I_{a+}^\alpha f(b)\right] \geq \frac{1}{\alpha} f\left(\frac{a+b}{2}\right).$$

On the other hand,

$$\int_0^1 \lambda^s(1-\lambda)^s\Big[f(\lambda a+(1-\lambda)b) + f((1-\lambda)a+\lambda b)\Big]d\lambda \leq [f(a)+f(b)]\int_0^1 [\lambda^s+(1-\lambda)^s]d\lambda.$$

Multiply through by $\lambda^{s\alpha-s}$ and integrate over $\lambda \in [0,1]$:

$$\int_0^1 \lambda^{s\alpha}(1-\lambda)^s\Big[f(\lambda a+(1-\lambda)b) + f((1-\lambda)a+\lambda b)\Big]d\lambda \leq [f(a)+f(b)]\int_0^1 [\lambda^{s\alpha}+\lambda^{s\alpha-s}(1-\lambda)^s]d\lambda.$$

Evaluating each of the integrals gives,

$$\begin{aligned}\int_0^1 \lambda^{s\alpha}(1-\lambda)^s f(\lambda a+(1-\lambda)b)d\lambda &= \int_b^a \left(\frac{b-x}{b-a}\right)^{s\alpha}\left(\frac{x-a}{b-a}\right)^s f(x)\frac{dx}{a-b}\\ &= \frac{1}{(b-a)^{s(\alpha+1)+1}}\int_a^b (b-x)^{s\alpha}(x-a)^s f(x)dx\\ &= \frac{\Gamma(s\alpha+1)}{(b-a)^{s(\alpha+1)+1}} I_{a+}^{s\alpha+1}\left[(b-a)^s f(b)\right].\end{aligned}$$

Also,

$$\begin{aligned}\int_0^1 \lambda^{s\alpha}(1-\lambda)^s f((1-\lambda)a+\lambda b)d\lambda &= \int_a^b \left(\frac{x-a}{b-a}\right)^{s\alpha}\left(\frac{b-x}{b-a}\right)^s f(x)\frac{dx}{b-a}\\ &= \frac{1}{(b-a)^{s(\alpha+1)+1}}\int_a^b (x-a)^{s\alpha}(b-x)^s f(x)dx\\ &= \frac{\Gamma(s\alpha+1)}{(b-a)^{s(\alpha+1)+1}} I_{b-}^{s\alpha+1}\left[(b-a)^s f(a)\right].\end{aligned}$$

Finally,

$$\begin{aligned}\int_0^1 [\lambda^{s\alpha}+\lambda^{s\alpha-s}(1-\lambda)^s]d\lambda &= \frac{1}{1+s\alpha} + \frac{\Gamma(1+s)\Gamma(1-s+s\alpha)}{\Gamma(2+s\alpha)}\\ &= \frac{\Gamma(1+s\alpha)+\Gamma(1+s)\Gamma(1-s(1-\alpha))}{(1+s\alpha)\Gamma(1+s\alpha)}.\end{aligned}$$

Combining the integrals together, we obtain

$$\frac{\Gamma(s\alpha+1)}{(b-a)^{s(\alpha+1)+1}}\Big[I_{b-}^{s\alpha+1}\left[(b-a)^s f(a) + I_{a+}^{s\alpha+1}\left[(b-a)^s f(b)\right.\right.\Big] \leq \frac{\Gamma(1+s\alpha)+\Gamma(1+s)\Gamma(1-s(1-\alpha))}{(1+s\alpha)\Gamma(1+s\alpha)}\left[f(a)+f(b)\right],$$

and therefore,

$$\frac{\Gamma^2(s\alpha+1)}{(b-a)^{s(\alpha+1)+1}\big[\Gamma(1+s\alpha)+\Gamma(1+s)\Gamma(1-s(1-\alpha))\big]}\left[I_{b^-}^{s\alpha+1}\big[(b-a)^s f(a)+I_{a^+}^{s\alpha+1}\big[(b-a)^s f(b)\right]\leq\frac{f(a)+f(b)}{1+s\alpha}.$$

□

Katugampola generalized the above integrals for functions $f \in X_c^p(a,b)$ as follows: Let $X_c^p(a,b)$, $c \in \mathbb{R}$, denote a set of complex valued Lebesgue measurable functions f on $[a,b]$ with the norm

$$\|f\|_{X_c^p}=\left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}<\infty,\ 1\leq p<\infty,$$

and

$$\|f\|_{X_c^\infty}=\sup ess_{x\in(a,b)}|t^c f(t)|.$$

Definition 8 ([12,13]). If $f \in X_c^p(a,b)$. Then the left (and respectively the right) Katugampola fractional integral of order $\alpha \geq 0$ is given by

$$^{\rho}I_{a^+}^{\alpha}f(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t (t^\rho-s^\rho)^{\alpha-1}s^{\rho-1}f(s)ds,\ \ t\in[a,b]$$

and

$$^{\rho}I_{b^-}^{\alpha}f(t)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_t^b (s^\rho-t^\rho)^{\alpha-1}s^{\rho-1}f(s)ds,\ \ t\in[a,b].$$

Katugampola gave a generalization of different fractional integrals as follows:

Definition 9 ([12,13]). Let $f \in X_c^p(a,b)$, $\alpha \geq 0$ and $\beta,\rho,\eta,\kappa \in \mathbb{R}$. Then the left (and respectively the right) fractional integrals of f is given by

$$^{\rho}I_{a^+,\eta,\kappa}^{\alpha,\beta}f(t)=\frac{\rho^{1-\beta}t^{\kappa}}{\Gamma(\alpha)}\int_a^t (t^\rho-s^\rho)^{\alpha-1}s^{\rho(\eta+1)-1}f(s)ds,\ \ 0\leq a<t<b\leq\infty$$

and

$$^{\rho}I_{b^-,\eta,\kappa}^{\alpha,\beta}f(t)=\frac{\rho^{1-\beta}t^{\rho\eta}}{\Gamma(\alpha)}\int_t^b (s^\rho-t^\rho)^{\alpha-1}s^{\kappa+\rho-1}f(s)ds,\ \ 0\leq a<t<b\leq\infty.$$

We generalize Inequality (5) as follows:

$$f(\lambda^{\tilde{\rho}}x^{\tilde{\rho}}+(1-\lambda^{\tilde{\rho}})y^{\tilde{\rho}})\leq\frac{f(x^{\tilde{\rho}})}{\lambda^{s\tilde{\rho}}}+\frac{f(y^{\tilde{\rho}})}{(1-\lambda^{\tilde{\rho}})^s},\tag{12}$$

with $\tilde{\rho}=\rho(\eta+1)$.

Theorem 4. *Let $f \in X_c^p(a^{\tilde{\rho}},b^{\tilde{\rho}})$. Suppose $f \in Q_s(I)$ with $I=[a^{\tilde{\rho}},b^{\tilde{\rho}}]$ and satisfies (12). Then*

$$f\left(\frac{a^{\tilde{\rho}}+b^{\tilde{\rho}}}{2}\right)\leq\frac{2^s\Gamma(\alpha+1)}{\rho^{1-\beta}}\frac{\tilde{\rho}}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})^{\alpha}}\left[\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I_{b^{\tilde{\rho}-},\eta,\kappa}^{\alpha,\beta}f(a^{\tilde{\rho}})+\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I_{a^{\tilde{\rho}+},\eta,\rho\eta}^{\alpha,\beta}f(b^{\tilde{\rho}})\right],$$

and

$$\frac{1}{\rho^{1-\beta}}\frac{\Gamma^2\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})^{s(\alpha+1)+\frac{1}{\tilde{\rho}}}}\frac{1}{\Gamma\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)+\Gamma(1+s)\Gamma\left(\frac{1+s\tilde{\rho}(\alpha-1)}{\tilde{\rho}}\right)}$$
$$\times\left[\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I_{b^{\tilde{\rho}-},\eta,\kappa}^{s\alpha+\frac{1}{\tilde{\rho}},\beta}\big[(b^{\tilde{\rho}}-a^{\tilde{\rho}})^s f(a^{\tilde{\rho}})\big]+\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I_{a^{\tilde{\rho}+},\eta,\rho\eta}^{s\alpha+\frac{1}{\tilde{\rho}},\beta}\big[(b^{\tilde{\rho}}-a^{\tilde{\rho}})^s f(b^{\tilde{\rho}})\big]\right]\leq\frac{f(a^{\tilde{\rho}})+f(b^{\tilde{\rho}})}{1+s\alpha\tilde{\rho}},$$

where $\tilde{\rho}:=\rho(\eta+1)$.

Proof. For $\lambda^{\tilde{\rho}} = \frac{1}{2}$, $x^{\tilde{\rho}} = t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}}$, $y^{\tilde{\rho}} = (1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}}$, we have

$$2^s\left[f(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}}) + f((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}})\right] \geq f\left(\frac{a^{\tilde{\rho}}+b^{\tilde{\rho}}}{2}\right).$$

Multiply through by $t^{\alpha\tilde{\rho}-1}$, $\alpha, \tilde{\rho} > 0$, and integrating over t in the interval $[0,1]$ to obtain:

$$2^s\left[\int_0^1 t^{\alpha\tilde{\rho}-1} f(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}})dt + \int_0^1 t^{\alpha\tilde{\rho}-1} f((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}})dt\right] \geq \frac{1}{\alpha\rho(\eta+1)} f\left(\frac{a^{\tilde{\rho}}+b^{\tilde{\rho}}}{2}\right).$$

Evaluating the first integral, we let $t^{\tilde{\rho}} = \frac{b^{\tilde{\rho}}-x^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}$ and $\frac{x^{\tilde{\rho}-1}}{x^{\tilde{\rho}}-b^{\tilde{\rho}}}dx = \frac{1}{t}dt$, to have

$$\begin{aligned}\int_0^1 t^{\alpha\tilde{\rho}-1} f(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}})dt &= \int_0^1 t^{\alpha\tilde{\rho}} f(t^{\tilde{\rho}}a^{\tilde{\rho}} + (1-t^{\tilde{\rho}})b^{\tilde{\rho}})t^{-1}dt \\ &= \int_b^a \left(\frac{b^{\tilde{\rho}}-x^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)^{\alpha} f(x^{\tilde{\rho}}) \frac{x^{\tilde{\rho}-1}}{x^{\tilde{\rho}}-b^{\tilde{\rho}}}dx \\ &= \frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)^{\alpha}} \int_a^b \frac{x^{\tilde{\rho}-1}}{\left(b^{\tilde{\rho}}-x^{\tilde{\rho}}\right)^{1-\alpha}} f(x^{\tilde{\rho}})dx \\ &= \frac{\Gamma(\alpha)}{\rho^{1-\beta}(b^{\tilde{\rho}})^{\rho\eta}} \frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)^{\alpha}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta} f(b^{\tilde{\rho}}).\end{aligned}$$

Similarly, for the second integral, we obtain:

$$\int_0^1 t^{\alpha\tilde{\rho}-1} f((1-t^{\tilde{\rho}})a^{\tilde{\rho}} + t^{\tilde{\rho}}b^{\tilde{\rho}})dt = \frac{\Gamma(\alpha)}{\rho^{1-\beta}(a^{\tilde{\rho}})^{k}} \frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)^{\alpha}} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa} f(a^{\tilde{\rho}}).$$

Combine the two integrals to give,

$$\frac{2^s\Gamma(\alpha)}{\rho^{1-\beta}} \frac{\alpha\rho(\eta+1)}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)^{\alpha}}\left[\frac{1}{(a^{\tilde{\rho}})^k} {}^{\rho}I^{\alpha,\beta}_{b^{\tilde{\rho}-},\eta,\kappa} f(a^{\tilde{\rho}}) + \frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{\alpha,\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta} f(b^{\tilde{\rho}})\right] \geq f\left(\frac{a^{\tilde{\rho}}+b^{\tilde{\rho}}}{2}\right).$$

To prove the second part of the theorem, we start with the following

$$\lambda^{s\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s f(\lambda^{\tilde{\rho}}a^{\tilde{\rho}} + (1-\lambda^{\tilde{\rho}})b^{\tilde{\rho}}) \leq (1-\lambda^{\tilde{\rho}})^s f(a^{\tilde{\rho}}) + \lambda^{s\tilde{\rho}} f(b^{\tilde{\rho}}), \tag{13}$$

and

$$\lambda^{s\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s f((1-\lambda^{\tilde{\rho}})a^{\tilde{\rho}} + \lambda^{\tilde{\rho}}b^{\tilde{\rho}}) \leq \lambda^{s\tilde{\rho}} f(a^{\tilde{\rho}}) + (1-\lambda^{\tilde{\rho}})^s f(b^{\tilde{\rho}}). \tag{14}$$

First add (13) and (14); multiply it by $\lambda^{s\alpha\tilde{\rho}-s\tilde{\rho}}$ and integrate over $\lambda \in [0,1]$ to give

$$\begin{aligned}&\int_0^1 \lambda^{s\alpha\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s\Big[f(\lambda^{\tilde{\rho}}a^{\tilde{\rho}} + (1-\lambda^{\tilde{\rho}})b^{\tilde{\rho}}) + f((1-\lambda^{\tilde{\rho}})a^{\tilde{\rho}} + \lambda^{\tilde{\rho}}b^{\tilde{\rho}})\Big]d\lambda \\ &\leq [f(a^{\tilde{\rho}}) + f(b^{\tilde{\rho}})]\int_0^1 [\lambda^{s\alpha\tilde{\rho}} + \lambda^{s\alpha\tilde{\rho}-s\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s]d\lambda.\end{aligned} \tag{15}$$

To evaluate the first integral, let $\lambda^{\tilde{\rho}} = \frac{b^{\tilde{\rho}}-x^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}$, $\frac{x^{\tilde{\rho}-1}}{x^{\tilde{\rho}}-b^{\tilde{\rho}}}\left(\frac{b^{\tilde{\rho}}-x^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)^{\frac{1}{\tilde{\rho}}}dx = d\lambda$ and thus,

$$\begin{aligned}\int_0^1 \lambda^{s\alpha\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s f(\lambda^{\tilde{\rho}}a^{\tilde{\rho}} + (1-\lambda^{\tilde{\rho}})b^{\tilde{\rho}})d\lambda &= \int_b^a \left(\frac{b^{\tilde{\rho}}-x^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)^{s\alpha}\left(\frac{x^{\tilde{\rho}}-a^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)^{s} f(x^{\tilde{\rho}})\frac{x^{\tilde{\rho}-1}}{x^{\tilde{\rho}}-b^{\tilde{\rho}}}\left(\frac{b^{\tilde{\rho}}-x^{\tilde{\rho}}}{b^{\tilde{\rho}}-a^{\tilde{\rho}}}\right)^{\frac{1}{\tilde{\rho}}}dx \\ &= \frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)^{s\alpha+s+\frac{1}{\tilde{\rho}}}}\int_a^b \left(b^{\tilde{\rho}}-x^{\tilde{\rho}}\right)^{s\alpha-1+\frac{1}{\tilde{\rho}}}(x^{\tilde{\rho}}-a^{\tilde{\rho}})^s x^{\tilde{\rho}-1} f(x^{\tilde{\rho}})dx \\ &= \frac{\Gamma(s\alpha+\frac{1}{\tilde{\rho}})}{\rho^{1-\beta}} \frac{1}{\left(b^{\tilde{\rho}}-a^{\tilde{\rho}}\right)^{s\alpha+s+\frac{1}{\tilde{\rho}}}} \frac{1}{(b^{\tilde{\rho}})^{\rho\eta}} {}^{\rho}I^{s\alpha+\frac{1}{\tilde{\rho}},\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}\left[(b^{\tilde{\rho}}-a^{\tilde{\rho}})^s f(b^{\tilde{\rho}})\right].\end{aligned}$$

For the second integral, we follow the same procedure to arrive at

$$\int_0^1 \lambda^{s\alpha\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s f((1-\lambda^{\tilde{\rho}})a^{\tilde{\rho}}+\lambda^{\tilde{\rho}}b^{\tilde{\rho}})d\lambda = \frac{\Gamma(s\alpha+\frac{1}{\tilde{\rho}})}{\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})^{s\alpha+s+\frac{1}{\tilde{\rho}}}}\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I^{s\alpha+\frac{1}{\tilde{\rho}},\beta}_{b^{\tilde{\rho}-},\eta,\kappa}\left[(b^{\tilde{\rho}}-a^{\tilde{\rho}})^s f(a^{\tilde{\rho}})\right].$$

Evaluating the integral on the right hand side of (15), we have

$$\begin{aligned}\int_0^1 [\lambda^{s\alpha\tilde{\rho}}+\lambda^{s\alpha\tilde{\rho}-s\tilde{\rho}}(1-\lambda^{\tilde{\rho}})^s]d\lambda &= \frac{1}{1+s\alpha\tilde{\rho}}+\frac{\Gamma(1+s)\Gamma\left(\frac{1+s\alpha\tilde{\rho}-s\tilde{\rho}}{\tilde{\rho}}\right)}{\tilde{\rho}\Gamma\left(\frac{1+\tilde{\rho}+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)}\\ &= \frac{1}{1+s\alpha\tilde{\rho}}+\frac{\Gamma(1+s)\Gamma\left(\frac{1+s\tilde{\rho}(\alpha-1)}{\tilde{\rho}}\right)}{(1+s\alpha\tilde{\rho})\Gamma\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)}\\ &= \frac{\Gamma\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)+\Gamma(1+s)\Gamma\left(\frac{1+s\tilde{\rho}(\alpha-1)}{\tilde{\rho}}\right)}{(1+s\alpha\tilde{\rho})\Gamma\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)}.\end{aligned}$$

Now, combine all the integrals of (15) together to obtain

$$\begin{aligned}&\frac{\Gamma(s\alpha+\frac{1}{\tilde{\rho}})}{\rho^{1-\beta}}\frac{1}{(b^{\tilde{\rho}}-a^{\tilde{\rho}})^{s\alpha+s+\frac{1}{\tilde{\rho}}}}\left[\frac{1}{(a^{\tilde{\rho}})^k}{}^{\rho}I^{s\alpha+\frac{1}{\tilde{\rho}},\beta}_{b^{\tilde{\rho}-},\eta,\kappa}\left[(b^{\tilde{\rho}}-a^{\tilde{\rho}})^s f(a^{\tilde{\rho}})\right]+\frac{1}{(b^{\tilde{\rho}})^{\rho\eta}}{}^{\rho}I^{s\alpha+\frac{1}{\tilde{\rho}},\beta}_{a^{\tilde{\rho}+},\eta,\rho\eta}\left[(b^{\tilde{\rho}}-a^{\tilde{\rho}})^s f(b^{\tilde{\rho}})\right]\right]\\ &\leq \frac{\Gamma\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)+\Gamma(1+s)\Gamma\left(\frac{1+s\tilde{\rho}(\alpha-1)}{\tilde{\rho}}\right)}{(1+s\alpha\tilde{\rho})\Gamma\left(\frac{1+s\alpha\tilde{\rho}}{\tilde{\rho}}\right)}[f(a^{\tilde{\rho}})+f(b^{\tilde{\rho}})].\end{aligned}$$

□

3. Conclusion

The results focus on new generalized fractional Hadamard type inequalities for s–Godunova–Levin functions of the second kind. The obtained results generalize and extend already existing results.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability: No data is required for this research.

Funding Information: No funding is available for this research.

Acknowledgments: The first author acknowledges the continuous support of the University of Hafr Al Batin, Saudi Arabia. We also extend our appreciation to the anonymous referee(s) for a job thoroughly well done.

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