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Supply Quantitative Model à la Leontief*

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Abstract

This paper focuses on the supply quantitative model system of input–output, which is equivalent to the demand quantitative model system of Leontief. This model allows us to define the total supplied quantities of commodities for any given supplied quantity of primary factors, and consequently enables us to define the final uses of commodities. The supply quantitative model is based on the direct output coefficients of primary factors. The Hadamard Product is also used. The quantitative supply system models might be useful tools in planning the economics of countries that have higher unemployment of primary factors, especially labour.

Keywords: Leontief, Demand and Supply Quantitative Model, Output Coefficients, the Hadamard Product

1. Introduction

Leontief used the term "Input-Output" in the title of his first and seminal paper on Input-Output Analysis, in "Quantitative Input and Output Relations in the Economic System of the United States" [1]. This means that every activity in economics is simultaneously characterized by two sides: income (revenue) expenditure, demand supply, input-output, export-import, and so on. In other words, the income of a certain economic unit (household, firm, institution, country) is concurrently expenditure for another economic unit; demand for any commodity by any individual is also supply for another individual or firm; input of any commodity to a certain sector is also output for the sector producing that commodity. Using this postulate, Leontief describes his input-output table as: 'Each row contains the revenue (output) items of one separate business (or household) … If read vertically, column by column, the table shows the expenditure sides of the successive accounts' [1]. Therefore, this allows us to describe and analyze economies on two sides so that if the same conditions exist, the results must be equivalent for both—in quantity and price terms. For example, the development of the whole economy might be modeled on either the input side or the output side. Each direction has its own targets and allows us to solve different types of problems of contemporary economics.

Since that period in economic literature on Input-Output, there have been attempts to formulate models describing the whole economy in both sides: demand (input) and supply (output) for quantity and supply (input) and demand (output) for price. Until today, only two types of system models of input-output have been formulated. These system models were first formulated by Leontief in their original form, and in the following years they were improved upon: quantitative demand (input) and price supply (input) models ([1-4]). The first model allows us to define the demand (required) quantity of the total production (input) of commodities for any given amount of final uses and consequently also for defining the demand (required) quantity of the primary factors. The second model allows us to define the cost of production (supply price) of commodities on the basis of primary factors' prices which are determined according to their required quantities by means of their total supply curves ([5,6]).

When Ghosh ([7,8]) formulated the allocation model, it was unfortunately labeled into an "output" (supply, supply-driven) model by his followers ([9-12])[1]. Moreover, Dietzenbacher ([10]; see also [13]) attempted to prove that Ghosh's allocation model is equivalent to Leontief's price model. However, the recent paper [6]

*The author thanks Prof. A. Brody and Prof. E. Einy for useful suggestions; *This paper is dedicated to Leontief's* 100*th* *birthday, and* 70 *years since his first paper on Input-Output.*

[1]I also, unfortunately, finally called Ghosh's model an output model, despite the fact that in my first paper [19] the distribution and output coefficients (are) were used equivalently, but in my book [5] I mentioned 'distribution coefficients' only once and after that 'output coefficients' and 'output models' were used.

shows that Leontief's Input-Output system model differs from Ghosh's system, and therefore they cannot be equivalent.

This paper focuses on the supply quantitative model that allows us to define the total supplied quantities of commodities for any given supplied quantity of primary factors, and consequently to define the final uses of commodities for both physical and monetary input-output systems. The supply quantitative model à la Leontief is based on the output coefficients of primary factors and input coefficients of commodities. The output coefficients of primary factors are the inverse magnitudes of their inputs coefficients; therefore, if the input coefficients are given and constant by assumptions, then the output coefficients are also given and constant. The Hadamard Product is also used.

Hence, this model allows us to define the total supplied quantities of commodities for any given supplied quantity of primary factors, and consequently to define the final uses of commodities. Such approach allows us to manipulate by each components of primary factor (types of labour or fixed capital). While Ghosh's model is based on the allocation coefficients of commodities, which are not inverted of the input coefficients, and on the input coefficients of primary factors, and therefore, allows manipulate generally by an aggregate magnitude of value added.

This paper consists of two sections. Following the introduction the first section describes supply quantitative equilibrium system models for Input-Output in physical terms; and the second section deals with supply quantitative equilibrium system models for Input-Output in monetary terms. Finally conclusions are presented.

2. Supply Quantitative Equilibrium for I-O in Physical Terms à la Leontief[2]

Let us start with the demand quantitative model of Leontief's input-output system, before describing the supply quantitative model, for two reasons: (1) to consider additional property of the demand model; and (2) to compare and understand characteristics of these models. The demand quantitative equilibrium for I-O in physical terms consists of two systems [6]:

$$\underline{x}^d = \underline{A}\left(\underline{x}^d\right) + \underline{y}^d, \text{or } \underline{x}^d = \left(I - \underline{A}\right)^{-1} \underline{y}^d, \text{or } \underline{x}^d = \underline{B}\underline{y}^d \quad (1.1)$$

$$\begin{gathered}\underline{v}^d = \underline{V}i_n = \left(\underline{C}\hat{\underline{x}}^d\right)i_n = \underline{C}\underline{x}^d \le \underline{v}_0 \\ \text{or } \underline{v}^d = \underline{C}\underline{x}^d = \underline{C}\underline{B}\underline{y}^d \le \underline{v}_0\end{gathered} \quad (1.2)$$

where

$\underline{X} - \left[\underline{x}_{ij}\right]$—is the square matrix ($n$*$n$) of the quantitative flows of commodities in the production;

$\underline{Y} = \left[\underline{y}_{ir}\right]$—is the matrix ($n$*$R$) of the quantitative flows of commodities to the categories of final uses;

$\underline{y}^d$ – is the column vector (n*1) of commodities' quantities for final uses;

$\underline{x}^d$ —is the column vector (n*1) of the total output quantity of commodities;

$\underline{V} - \left[\underline{v}_{kj}\right]$—is the matrix ($m$*$n$) of the quantitative flows of primary factors to the sectors of production;

$\underline{v}^d$ – is the column vector (m*1) of the total quantities of primary factors required in the production;

$\underline{A} - \left[\underline{a}_{ij}\right]$—is the square matrix ($n$*$n$) of the direct input coefficients of commodities in real (physical) terms in the production and

$$\underline{A} = \underline{X}\left(\hat{\underline{x}}^d\right)^{-1}, i.e., \ \underline{a}_{ij} = \frac{\underline{x}_{ij}}{\underline{x}_j^d}; \quad (1.3)$$

i.e., the input coefficient $\underline{a}_{ij}$ measure quantity of commodity i required for the production of one unit of commodity j in physical terms;

$\underline{C} - \left[\underline{c}_{kj}\right]$—is the matrix ($m$*$n$) of the direct input coefficients of factors in real physical terms in the production and

$$\underline{C} = \underline{V}\left(\hat{\underline{x}}^d\right)^{-1}, i.e., \ \underline{c}_{kj} = \frac{\underline{v}_{kj}}{\underline{x}_j^d}; \quad (1.4)$$

i.e., the input coefficient of primary factors $\underline{c}_{kj}$ measure quantity of factor k required for the production of one unit of commodity j in physical terms;

$\underline{B}$ —is Leontief's inverse matrix, and $\underline{b}_{ij}$ —is the total required quantities (direct and indirect inputs) of commodity i to a satisfied one unit of demand of the commodity j;

$\underline{v}_0$ —is the vector of the available quantities of primary factors;

i_n —is a unit column vector (n*1);

The system (1.1) allows us to obtain the total required quantities of commodities for any given quantities of final uses for the certain conditions of the direct input coefficients of commodities $\underline{A}$. Consequently, by the substitution of the obtained required output quantities in the system (1.2), the required quantities of primary factors are defined as $\underline{v}^d$. Therefore, if the required quantities of primary factors are within the limit quantities drawing from their supply curves, *i.e.*, if the required quantities are less or equal to the available quantities ($\underline{v}^d \le \underline{v}_0$), then there is a quantitative equilibrium and then a price equilibrium establishing might be considered.

[2]Input-Output in physical terms, in this paper, differs from the physical input-output table (PIOT), which recently appeared in input-output literature. In the former, each commodity has its own physical measurement: meter, ton, unit, M^3 and so on, while in the latter, all commodities have uniform physical measurement, for example ton.

Conversely, when, if at least the required quantity for one factor is larger than its available quantity, then the process must be carried out for the new different quantities for final uses, until the above conditions are satisfied.

Worthy to discuss the character of changes of the total required quantities of primary factors due to change of quantities of final uses. We assume, for the simplification, that only the quantity of final use for a one of sector (commodity) l is changed (increased) ($\underline{y}_l^d$), while the final uses for other sectors (commodities) stay unchanged. Substitute this in (1.1), and we have:

$$\underline{x}^{d1} = \underline{B}\underline{y}^{d1} = \underline{B}\left(\underline{y}^d + \Delta\underline{y}^d\right) = \underline{x}^d + \Delta\underline{x}^d \quad (1.5)$$

where $\Delta\underline{y}^d$—is the column vector (n*1) all components of which are zero except of the component l that equal to $\Delta\underline{y}_l^d$. So

$$\Delta\underline{x}^d = \underline{B}\Delta\underline{y}^d \quad (1.6)$$

and

$$\Delta\underline{x}_i^d = \underline{b}_{ij}\Delta\underline{y}_l^d,\ j = l,\ \left(i = 1, 2, \cdots n\right) \quad (1.7)$$

From (1.7) we can conclude that increasing the final use of the commodity of a certain sector $\boldsymbol{l}$ either increases the total production of commodities of the sectors where according inverse coefficients of inputs are more than zero ($\underline{b}_{ij} > 0$, $j = \boldsymbol{l}$) or unchanged if according inverse coefficients of inputs equal zero ($\underline{b}_{ij} = 0$, $j = \boldsymbol{l}$). Consequently, the quantities of primary factors are either increased, if direct input coefficients of primary factors are more than zero ($\underline{c}_{kj} > 0$), or unchanged if direct input coefficients of primary factors are equal to zero ($\underline{c}_{kj} = 0$) in the sectors where the total production is increased. This is

$$\Delta\underline{v}_{kj} = \underline{c}_{kj}\underline{b}_{j(=i)l}\Delta\underline{y}_l^d,\ \left(k = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n\right) \quad (1.8)$$

Now, the total increase of each primary factor is determined as

$$\Delta\underline{v}_k^d = \Sigma_{j=1}^n \Delta\underline{v}_{kj} = \Sigma_{j=1}^n \underline{c}_{kj}\underline{b}_{j(=i)l}\Delta\underline{y}_l^d,\ \left(k = 1, 2, \cdots, m\right) \quad (1.9)$$

From (1.9) $\Delta\underline{y}_l^d$ is determined as

$$\Delta\underline{y}_l^d = \Delta\underline{v}_k^d \Big/ \Sigma_{j=1}^n \underline{c}_{kj}\underline{b}_{j(=i)l} \quad (1.10)$$

However, these total increases of each primary factor must be less or equal to its unemployed quantities, this is:

$$\Delta\underline{v}_k^d \le \underline{v}_{k0} - \underline{v}_k^d,\ \left(k = 1, 2, \cdots, m\right) \quad (1.11)$$

Therefore

$$\max\Delta\underline{y}_l^d = \min_{1\le k\le m}\left(\underline{v}_{k0} - \underline{v}_k^d\right) \Big/ \sum_{j=1}^n \underline{c}_{kj}\underline{b}_{j(=i)l}, \quad (1.12)$$

This proves the following theorem:

Theorem 1 If matrix $\underline{A}$ is positive ($\underline{A} \ge 0$) and productive ($\underline{x} > \underline{xA}$), and if the quantity of final use of a certain sector $\underline{y}_l^d$ is increased and final uses for all other sectors are unchanged, then the required quantities of primary factors are either increased if direct input coefficients of primary factors are more than zero ($\underline{c}_{kj} > 0$) or unchanged if direct input coefficients of primary factors are equal to zero ($\underline{c}_{kj} = 0$) for the sectors where the total production is increased; also the magnitude of the increase of final use of a certain sector (commodity) is limited by the unemployed supply quantities of primary factors (1.12).

To sum up, this theorem indicates that increasing of the quantity in the final use of the commodity of a certain sector, increases the required quantities of primary factors almost in all sectors.

On the other hand, careful examination of the demand system models shows that they might be used for opposite purposes (direction) too. Namely, the system (1.1) might be used to obtain the total quantities of final uses for any given total quantity of commodities, rewriting it as:

$$\underline{y}^d = \left(I - \underline{A}\right)\underline{x}^d \quad (1.13)$$

So, (1.13) allows us to obtain the total quantity of final uses for a given total quantities of commodities. This means that in order to determine the total demand of final uses, the total quantities of commodities have to be known, so that the latter have to be connected with primary factors. For example, the total quantities of commodities must be determined on the basis of the given quantities of primary factors. In other words, the opposite model to (1.2) is required.

The question is, therefore, whether the system (1.2) may be transformed into such a model which may allow us to determine the total quantities of commodities for any given quantities of primary factors. Until today, the answer was obviously negative. It asserted that the column of primary factors for a certain sector, for the input-output system in physical terms, is heterogeneous and therefore, not be summed. Thus the negative answer is based on the ordinary analysis of input-output system models.

Let us try another approach.

Let's start from the determination of the flows of primary factors to sectors of production ($\underline{V}$—matrix). From (1.2) it is determined that:

$$\underline{V}^d = \underline{C}\hat{\underline{x}}^d \quad (1.14)$$

If we take into account the fact that when regular matrix is multiplied on a diagonal matrix, it means that the first component of each row of regular matrix is multiplied on the element of the first column of the diagonal matrix and the second element of each row is multiplied on the element of the second column, and so on. Therefore, the diagonal matrix may be replaced by a matrix where all elements of a certain column are identical and equal to the according diagonal magnitude; and new matrix's dimension is defined according to the dimension of matrix $\underline{C}$, *i.e*. (m*n). This is, taking case $\hat{\underline{x}}^{d}$ under discussion, might be replaced by the matrix $\underline{X}^{dd}$ (m*n) where all elements of the first column would be the total output of the first sector, all elements of the second column – the total output of the second sector, and so on:

$$\underline{X}^{dd}=\begin{pmatrix} \underline{x}^{d11} & \underline{x}^{d12} & \cdots & \underline{x}^{d1n} \\ \underline{x}^{d21} & \underline{x}^{d22} & \cdots & \underline{x}^{d2n} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{x}^{dm1} & \underline{x}^{dm2} & \cdots & \underline{x}^{dmn} \end{pmatrix} \quad (1.15)$$

It is necessary to emphasize that there might be an opposite case, namely, when a diagonal matrix is multiplied by a regular matrix, and, in such a case, each element of the row of replacing matrix has to be identical and the dimension of the matrix must be according to the regular matrix (vide infra).

Now, (1.14) might be rewritten as

$$\underline{V}^{d}=\underline{C}\hat{\underline{x}}^{d}=\underline{C}\circ\underline{X}^{dd} \quad (1.16)$$

The sign ($\circ$) means the *Hadamard product* of two matrices $\underline{C}$ and $\underline{X}^{dd}$ when matrix $\underline{V}^{d}$ is formed by the elementwise multiplication of their elements. The matrices must be the same size. So, every component of $\underline{V}^{d}$ is obtained as the following: each component of matrix $\underline{C}$ is multiplied on the according component of matrix $\underline{X}^{dd}$, for example, the element $\underline{c}_{23}$ is multiplied on the according element $\underline{x}^{d23}$.

On the other hand, from (1.16) $\underline{X}^{dd}$ might be determined as

$$\underline{X}^{dd}=\underline{V}^{d}\circ\underline{C}^{o} \quad (1.17)$$

where $\underline{C}^{o}$—is the matrix of direct output coefficients of primary factors, which are inverted of the direct input coefficients of primary factors and it is the same size and structure of the matrix $\underline{C}$, this is, $\underline{c}_{kj}^{o}=1/\underline{c}_{kj}$ if $\underline{c}_{kj}\gg 0$ and if $\underline{c}_{kj}=0$ then $\underline{c}_{kj}^{o}$ also equal to 0; the output coefficient indicates the quantities of commodity *j* produced by a unit of primary factor *k*.

If, by assumption, the direct input coefficients of primary factors are given and constant, then the direct output coefficients would also be given and constant. Therefore, according to (1.17) in order to determine the total quantities of commodities, the flow of primary factors to sectors (matrix $\underline{V}$) is required. As mentioned above for the equilibrium state, when it is determined from the demand side, the elements of a certain column of $\underline{X}^{dd}$ are identical, and they are the same quantity. But, when the elements of matrix $\underline{V}$ are determined accidentally as supply (notate as $\underline{V}^{s}$), according to the available quantities of primary factors, and they have to use for determination of the total output of commodities, then the total quantity of a certain commodity may be different for various primary factors. In such a case, it is necessary to choose one amount from them (vide infra).

The required quantities of primary factors ($\underline{v}^{d}$-column vector), which are determined by the required flows of primary factors to sectors of production ($\underline{V}^{d}$), has to be a source for the determination of the supplied version of the latter matrix ($\underline{V}^{s}$). If the required quantities of primary factors are far from their available quantities ($\underline{v}^{d}<\underline{v}_{0}$), then there are unemployed quantities of primary factors (including labour). Therefore, in such a situation, the opposite process is desirable, namely, the process has to start from the side of primary factors instead of the side of final uses as in the previous case. Here, in the beginning, the amount of quantities of primary factors (notate as $\underline{v}^{ds}$—the total supply quantities of primary factors) are determined and then their distribution between the sectors of production must be determine. So, the question now is how the given quantities of primary factors have to be distributed between sectors of production.

There are infinite ways of distribution of the given supply quantities of primary factors between production sectors, starting from the occasional distribution and finishing with the planning distribution according to a certain criterion. Let us discuss the type of distribution where the structure of new distribution is identical to the structure of the distribution for the demand side. For the purpose of defining the structure of the demand side let us rewrite the equation system (1.2) as follows:

$$\underline{v}_{k}^{d}=\underline{v}_{k1}^{d}+\underline{v}_{k2}^{d}+\cdots+\underline{v}_{kn}^{d},\left(k=1,2,\cdots,m\right) \quad (1.18)$$

or

$$\underline{v}_{k}^{d}=\underline{v}_{k1}^{d}\frac{\underline{v}_{k}^{d}}{\underline{v}_{k}^{d}}+\underline{v}_{k2}^{d}\frac{\underline{v}_{k}^{d}}{\underline{v}_{k}^{d}}+\cdots+\underline{v}_{kn}^{d}\frac{\underline{v}_{k}^{d}}{\underline{v}_{k}^{d}},\left(k=1,2,\cdots,m\right) \quad (1.19)$$

and

$$\underline{v}_{k}^{d}=\alpha_{k1}\underline{v}_{k}^{d}+\alpha_{k2}\underline{v}_{k}^{d}+\cdots+\alpha_{kn}\underline{v}_{k}^{d},\left(k=1,2,\cdots,m\right) \quad (1.20)$$

where

$$\alpha_{kj}=\frac{\underline{v}_{kj}^{d}}{\underline{v}_{k}^{d}},\left(k=1,2,\cdots,m;\ j=1,2,\cdots,n\right) \quad (1.21)$$

$$\sum_{j=1}^{n}\alpha_{kj}=1,\left(k=1,2,\cdots,m\right) \quad (1.22),$$

α_{kj}—is the share of the sector *j* in the total required quantities of primary factor *k*.

From (1.21) we can define

$$\underline{v}_{kj}^{d} = \alpha_{kj}\underline{v}_{k}^{d}, (k = 1,2,\cdots,m;\ j = 1,2,\cdots,n) \quad (1.23)$$

Therefore

$$\underline{V}^{d} = \alpha \circ \underline{V}^{dd}, \quad (1.24)$$

where $\circ$—is a sing of the *Hadamard product*;

α – $[\alpha_{kj}]$ —is the matrix (m*n) of distribution of primary factors between sectors of production;

$\underline{V}^{dd}$—is the matrix (m*n) where all elements of a certain row are identical (vide supra) and equal to the required quantity of the according factor.

So, assuming that α is constant (1.21) allows us to determine $\underline{V}^{d}$ when $\underline{V}^{dd}$ is given, that is, determine $\underline{V}^{s}$ when $\underline{V}^{ss}$ ($\underline{v}^{ds}$) is given.

To sum up, the process is completed. If the total supply quantities of primary factors are given then (1.24) allows us to determine their distribution between branches of production; substituting the obtain results into (1.17), the total supply quantities of commodities are obtained; thus, substituting the latter into (1.13), according quantities of the final uses of commodities are determined.

Therefore, assuming that the new total quantity of primary factors is $\underline{v}^{ds}$[3], the matrix $\underline{V}^{ss}$ is compiled where all elements of each row are the same according to $\underline{v}^{ds}$. Substituting it in (1.24), the matrix $\underline{V}^{s}$ is obtained. Namely:

$$\underline{V}^{s} = \alpha \circ \underline{V}^{ss} \quad (1.25).$$

Substitute the latter into (1.17) we have:

$$\underline{X}^{ss} = \underline{V}^{s} \circ \underline{C}^{o} \quad (1.26).$$

Because of that the total quantities of various primary factors are independently determined from the input structure of sectors, columns of the matrix $\underline{X}^{ss}$ might be heterogenic, and that is, components of a certain column might be different. So, there might be the following

$$\underline{X}^{ss} = \begin{pmatrix} \underline{x}^{s11} & \underline{x}^{s12} & \cdots & \underline{x}^{s1n} \\ \underline{x}^{s21} & \underline{x}^{s22} & \cdots & \underline{x}^{s2n} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{x}^{sm1} & \underline{x}^{sm2} & \cdots & \underline{x}^{smn} \end{pmatrix} \quad (1.27)$$

where $\underline{x}^{skj}$—is the total quantities of commodity j determined according to the supply quantities of primary factor k.

In such a situation, it is necessary to choose one component from each column according to the following criterion:

$$\underline{x}^{sj} = \min_{1\le k\le m}\left\{\underline{x}^{skj}\right\}, \ (j = 1,2,\cdots,n), \quad (1.28)$$

[3]Where (ds) expresses the fact that these quantities are determined from the supply side. Such notation is used in order not to confuse it with v^{s}–the total required inputs of all primary factors for a certain sector, using for the input-output systems in monetary terms.

This means that for each column the lowest total quantity is chosen to guarantee existence of required quantities of all primary factors.

Substitute these total quantities of commodities $\underline{x}^{s}$ into Equation (1.13) and the total quantities of final uses $\underline{y}^{ds}$ are obtained.

To sum up, the supply quantitative equilibrium for Input-Output in physical terms can be placed into the following systems:

$$\underline{X}^{ss} = \underline{V}^{s} \circ \underline{C}^{o}, \quad (1.29)$$

$$\underline{x}^{sj} = \min_{1\le k\le m}\left\{\underline{x}^{skj}\right\}, (j = 1,2,\cdots,n), \quad (1.30)$$

$$\underline{y}^{ds} = \left(I - \underline{A}\right)\left(\underline{x}^{s}\right)' \quad (1.31)$$

where $\underline{V}^{s}$, $\underline{C}^{o}$, $\underline{A}$—are given.

The Equation (1.29) defines the matrix of possible total products of commodities for each primary factor $\underline{X}^{ss}$ as the *Hadamard product* of the matrix of the flows of primary factors to sectors ($\underline{V}^{s}$) and the matrix of direct output coefficients of primary factors ($\underline{C}^{o}$). Here, there might be m different total quantities of commodity for each sector (commodity). Consequently, the equation system (1.30) allows for the choosing of one total quantity for each sector so that it might be possible from the point of all primary factors. Finally, the equation (1.31) allows obtaining the final uses of commodities for the choosing of total quantities of production.

From the point of using the supply quantitative model in practice is worthy to consider the character of changes of the total quantities of final uses in according to changing of primary factors. To simplify, assume that only the quantity of primary factors for one sector of production is changed, while other sectors are unchanged. This means that the total productions of the latter sectors are also unchanged.

Assume that the quantity of the primary factor k (labour) for the sector j is increased by $\Delta\underline{v}_{kj}^{s} > 0$ >. Substitute this in (1.26) we have

$$\underline{x}_{j}^{s1} = \left(\underline{v}_{kj}^{s} + \Delta\underline{v}_{kj}^{s}\right)\underline{c}_{kj}^{o} = \underline{x}_{j}^{s} + \Delta\underline{x}_{j}^{s} \succ \underline{x}_{j}^{s} \quad (1.32)$$

Assuming that such an increase of the total production of the commodity j is also possible from the side of other primary factors in this sector, *i.e.*, there exist unemployed quantities of the rest primary factors. This means that quantities of all primary factors are accordingly increased. Then, when we substitute the latter in the Equation (1.31) we have:

$$\begin{aligned} \underline{y}^{ds1} &= \left(I - \underline{A}\right)\underline{x}^{s1} = \left(I - \underline{A}\right)\left(\underline{x}^{s} + \Delta\underline{x}^{s}\right)' \\ &= \left(I - \underline{A}\right)\left(\underline{x}^{s}\right)' + \left(I - \underline{A}\right)\left(\Delta\underline{x}^{s}\right) = \underline{y}^{d} + \Delta\underline{y}^{ds} \end{aligned} \quad (1.33),$$

where $\Delta\underline{x}^s$—is the row vector (1*n) all components of which are zero except of the component i (= j) that equal to $\Delta\underline{x}_j^s$. So

$$\Delta\underline{y}^{ds} = \left(I - \underline{A}\right)\left(\Delta\underline{x}^s\right)' \quad (1.34).$$

Therefore

$$\Delta\underline{y}_l^{ds} = \left(1 - \underline{a}_{ii}\right)\Delta\underline{x}_j^s = \left(1 - \underline{a}_{ii}\right)\Delta\underline{v}_{kj}^s\underline{c}_{kj}^o, \quad \text{when } i = j \quad (1.35),$$

$$\Delta\underline{y}_l^{ds} = \left(-\underline{a}_{ij}\right)\Delta\underline{x}_j^s = \left(-\underline{a}_{ij}\right)\Delta\underline{v}_{kj}^s\underline{c}_{kj}^o, \quad \text{when } i \neq j, \left(i = 1, \cdots, i = j-1, i = j+1, \cdots, n\right) \quad (1.36).$$

From this we can conclude the following: (1) since $\underline{a}_{ii} \ll 1$ the final uses of sector i ($\underline{y}_i^{ds}$, when $i = j$) is increased by $\left(1 - a_{ii}\right)\Delta\underline{x}_j^s$; and (2) since a_{ij} (when $I \neq j$) might be either $\underline{a}_{ij} > 0$ or $\underline{a}_{ij} = 0$ the final uses of sector i ($\underline{y}_i^{ds}$, when $i \neq j$) either is decreased by $-a_{ii}\Delta\underline{x}_j^s$ or is not changed. Yet, the increase of the final use of the commodity in question cannot be more than its unsatisfied quantity, that is:

$$\Delta\underline{y}_i^{ds} \leq \underline{y}_{i0} - \underline{y}_i^d, \text{when } \left(i = j\right) \quad (1.37).$$

where $\underline{y}_{i0}$—is the maximum quantity of demand of the commodity i. And the decrease of the final uses of the other commodities cannot be more than their quantities of final use, that is

$$\Delta\underline{y}_i^{ds} \leq \underline{y}_i^d, \text{ when } (i \neq j) \quad (1.38)$$

Therefore, the largest magnitude of the increase of the primary factor is equal to the smallest magnitude between the increase of final use of the sector in question (see 1.35) and the decrease of final uses of other sectors (see 1.36):

$$\max \Delta\underline{v}_{kj}^s = \min_{1 \leq i \leq n}\left\{\left(\underline{y}_{i0} - \underline{y}_i^d\right)\Big/\left(1 - \underline{a}_{ii}\right)\underline{c}_{kj}^o\left(i = j\right); \underline{y}_j^d \Big/ \underline{a}_{ii}\underline{c}_{kj}^o\left(i \neq j\right)\right\} \quad (1.39)$$

By this the following theorem is proofed:

Theorem 2 If matrix $\underline{A}$ is positive ($\underline{A} \geq 0$) and productive ($\underline{x} > \underline{xA}$), and if quantities of all primary factors $\underline{v}_j^s$ of a certain sector j are increased by the same rate and primary factors for all other sectors are unchanged, then the final use of the sector in question $\underline{y}_i^{ds}$ ($i = j$) is increased and the final uses of other sectors $\underline{y}_i^{ds}$ ($i \neq j$) are either decreased when $\underline{a}_{ij} > 0$ or unchanged when $\underline{a}_{ij} = 0$ (when $i \neq j$); and the magnitude of the certain primary factor's (factors') increase in a certain sector is limited by the unsatisfied final uses of the sector in question and final uses of other sectors (1.39).

From the above we conclude that increasing the quantity of any primary factor for a certain sector, increase the final use of this sector and decrease or don't change the final uses of all other sectors.

To illustrate the suggested supply quantitative model of input-output, let us use Leontief's simplified input-output model ([14]; see also [15], while making two changes: first, instead of two types of Capital Stocks, only one type is considered; and second, Capital Stocks is measured in monetary terms instead of physical terms:

From this Table we can define the direct input coefficients of commodities and primary factors:

$$\underline{A} = \underline{X}\left(\hat{\underline{x}}^d\right)^{-1} = \begin{pmatrix} 25.0P & 20.0P \\ 14.0Y & 6.0Y \end{pmatrix}\begin{pmatrix} 1/100.0P & 0 \\ 0 & 1/50.0Y \end{pmatrix} = \begin{pmatrix} 0.25\,P/P & 0.4\,P/Y \\ 0.14\,Y/P & 0.12\,Y/Y \end{pmatrix}, \quad (1.40)$$

$$\underline{C} = \underline{V}\left(\hat{\underline{x}}^d\right)^{-1} = \begin{pmatrix} 250.0\$ & 350.0\$ \\ 55.0MH & 135.0MH \end{pmatrix}\begin{pmatrix} 1/100.0P & 0 \\ 0 & 1/50.0Y \end{pmatrix} = \begin{pmatrix} 2.5\$/P & 7.0\$/Y \\ 0.55\,MH/P & 2.7\,MH/Y \end{pmatrix} \quad (1.41)$$

where: P-Pounds, Y-Yards, MH-Man-Hours.

Assuming that the quantity of the final use of the second commodity is increased (by 10.0 Y) and the quantity of the final use of the first commodity is unchanged and they are equal to $(y^{d1})'$= (55.0P 40.0Y); then using (1.1a) we obtain according total output of commodities:

$$\underline{x}^{d1} = \left(I - \underline{A}\right)^{-1}\underline{y}^{d1} = \underline{B}\underline{y}^{d1} = \begin{pmatrix} 1.45\,P/P & 0.662\,P/Y \\ 0.232\,Y/P & 1.242\,Y/Y \end{pmatrix}\begin{pmatrix} 55.0P \\ 40.0Y \end{pmatrix} \cong \begin{pmatrix} 106.20P \\ 62.4.0Y \end{pmatrix} \quad (1.42).$$

Then $\underline{V}^{d1} = \underline{C}\hat{\underline{x}}^{d1} = \begin{pmatrix} 265.5.0\$ & 436.8.0\$ \\ 58.4MH & 168.5MH \end{pmatrix}$,

and

$$v^{d1} = V^{d1}i_n = \begin{pmatrix} 702.3\$ \\ 226.9MH \end{pmatrix} \quad (1.43).$$

Another assumption is that the available quantities of primary factors are $(\underline{v}_0)'$ = (800\$ 300MH). So, in comparison to required quantities $\underline{v}^{d1}$ and the available quantities, we can conclude that this is quantitative equilibrium and there are unemployed amounts of both primary factors. Yet, by comparing the new matrices of flows of primary factors to sectors $\underline{V}^{d1}$ with according matrix from

Table 1, we can also see that each element of the first is larger than the according element of the second, which is according to *Theorem* 1. This is because, in this case, all inverse input coefficients of commodities and all direct input coefficients of primary input are strictly positive (>0).

Now assuming that the goal of economics is to achieve full employment for both primary factors (the available quantity minus 3% for reserve), this is the new vector of suggested quantities of primary factors will be $(\underline{v}^{sd1})'$ = (776.0$ 281.0MH)'. For the following we need matrix α and $\underline{V}^{ss}$. The first might be computed on the basis of **Table 1**, and it is

$$\alpha = \begin{pmatrix} 0.417 & 0.583 \\ 0.29 & 0.71 \end{pmatrix} \tag{1.44}$$

and the second is

$$\underline{V}^{ss} = \begin{pmatrix} 776.0\$ & 776.0\$ \\ 281.0MH & 281.0MH \end{pmatrix} \tag{1.45}$$

And

$$\underline{V}^{s} = \alpha \circ \underline{V}^{ss} = \begin{pmatrix} 324.0\$ & 452.0\$ \\ 81.0MH & 200MH \end{pmatrix} \tag{1.46}$$

And, $\underline{C}^{o}$ the matrix of direct output coefficients of primary factors is

$$\underline{C}^{o} = 1/C = \begin{pmatrix} 0.4\,P/\$ & 0.143\,Y/\$ \\ 1.82\,P/MH & 0.37\,Y/MH \end{pmatrix} \tag{1.47}$$

Substitute (1.48) and (1.49) into (1.29) we obtain

$$\begin{aligned} \underline{X}^{ss} &= \underline{V}^{s} \circ \underline{C}^{o} \\ &= \begin{pmatrix} 324.0\$ & 452.0\$ \\ 81.0MH & 200MH \end{pmatrix} \times \begin{pmatrix} 0.4\,P/\$ & 0.143\,Y/\$ \\ 1.82\,P/MH & 0.37\,Y/MH \end{pmatrix} \\ &= \begin{pmatrix} 129.6P & 64.6Y \\ 147.4P & 74.0Y \end{pmatrix} \end{aligned}$$

(1.48).

We can see that the total output differs for various primary factors in both sectors and therefore, using the criterion of choice (1.30) we obtain (that) $\underline{x}^{s}$ = (129.6 P 64.6 Y). This means that the supply quantities of the first factor, Capital Stock, is fully employed, while the second factor, Labour, is not fully employed, here is its unemployed part. Therefore, in order to increase employment of the second factor it is necessary to increase the first factor, *i.e*. investment must be increased.

Finally, according quantity of final uses is determined by mean of (1.31), namely

$$\begin{aligned} \underline{y}^{d(s)} &= \left(I - \underline{A}\right)\left(\underline{x}^{s}\right)' \\ &= \begin{pmatrix} 0.75\,P/P & -0.4\,P/Y \\ -0.14\,Y/P & 0.88\,Y/Y \end{pmatrix} \begin{pmatrix} 129.6P \\ 64.6Y \end{pmatrix} \\ &= \begin{pmatrix} 71.4P \\ 38.8Y \end{pmatrix} \end{aligned} \tag{1.49}$$

Despite the fact that the total productions are increased in both sectors, (129.6P > 106.2P, and 64.6Y > 62.4Y), the final use of the first sector is increased (71.4P > 55.0P), however, the final use of the second sector is decreased (38.8Y < 40.0Y). These results are according to *Theorem* 2, because the rate of increase of the first sector is greater than the second sector (0.22 > 0.035) and therefore, the increasing of the final use of the second sector deriving from the increasing its total production (0.88 × 2.4 = 2.1) is less than the decreasing deriving from the increasing of the total production of the first sector (0.14 × 23.4 = 3.3). While, the increasing of the final use of the first sector deriving from the increasing its total production (0.75 × 23.4 = 17.55) is greater than the decreasing deriving from the increasing of the total production of the second sector (0.4 × 2.4 = 0.96).

For the clearly demonstration properties of the *Theorem* 2, assume that the whole unemployed quantity of the first factor is used in the second sector, *i.e*., $\underline{v}_{12}^{s}$ = 526.0$; and therefore, $\underline{x}_{2}^{s} = \underline{v}_{12}^{s} \times \underline{c}_{12}^{o} = 526.0\$ \times 0.143Y/\$$. Since, the total production of the first sector is unchanged (= 100.0P, see **Table 1**)), then the according final uses are:

$$\begin{aligned} \underline{y}^{d(s')} &= \left(I - \underline{A}\right)\left(\underline{x}^{s'}\right)' \\ &= \begin{pmatrix} 0.75\,P/P & -0.4\,P/Y \\ -0.14\,Y/P & 0.88\,Y/Y \end{pmatrix} \begin{pmatrix} 100.0P \\ 75.2Y \end{pmatrix} \\ &= \begin{pmatrix} 45.0P \\ 52.2Y \end{pmatrix} \end{aligned} \tag{1.50}$$

The final use of the first sector is decreased by 10.0P {= (–0.4 × 25.2) or (45.0P – 55.0P)}, and the final use of the second sector is increased 22.2Y {= (0.88 × 25.2) or (52.2Y – 30.0Y)}.

Table 1. Hypothetical input-output in physical terms

	Agriculture	Manufacturing	Households	Total
Agriculture	25.0 Pounds	20.0 Pounds	55.0 Pounds	100.0 Pounds
Manufacturing	14.0 Yards	6.0 Yards	30.0 Yards	50.0 Yards
Capital Stocks	250.0 $	350.0 $		600.0 $
Labor	55.0 Man-Hours	135.0 Man-Hours	40.0 Man-Hours	230.0 Man-Hours

3. Supply Quantitative Equilibrium for I-O in Monetary Terms à la Leontief

In practice it is not always possible to separate quantities and prices with objective and subjective reasons [16]. Hence, the results of economic activities are usually presented in monetary terms. Therefore, almost all existing empirical I-O are compiled in monetary terms since Leontief's first input-output system [1].

Empirical (Marxian-Leontievian) I-O is characterized by "quantity" in monetary terms [17]. This means that in these cases, prices and quantities are not separated and they are amalgamated into one element. Each element is included as quantity and prices. Therefore, empirical I-O has a uniform measurement for all parts: commodities, factors and categories of final uses, namely, money measure. On the one hand, this creates some problems when it's used for planning and analysis. On the other hand this allows extending a scope of analysis by the formulation of additional models. For example, as it was mentioned above, Ghosh formulated the allocation model which, unfortunately, was labeled into an "output" (supply, supply-driven) model by his followers ([9-12]). It is important to stress that it is impossible to formulate such models for the I-O in physical terms. This is due to the heterogeneous character of both the structure of the use of factors for the production of certain products and the structure of commodities for a certain category of final uses. Moreover, Dietzenbacher [10] has attempted to prove that Ghosh's allocation model is equivalent to Leontief's price model. But, the recent paper [6] shows that Leontief's Input-Output system model differs from Ghosh's system, therefore they cannot be equivalent.

At this point let us start from the demand quantitative equilibrium model in monetary terms, which is identical to quantitative equilibrium for I-O in physical terms and consists of two systems:

$$x^d = A\left(x^d\right) + y^d, \text{ or } x^d = \left(I - A\right)^{-1} y^d, \text{ or } x^d = By^d \tag{2.1}$$

$$v^d = Vi_n = \left(C\hat{x}^d\right)i_n = Cx^d \le v_0, \text{ or } v^d = Cx^d = CBy^d \le v_0, \tag{2.2}$$

All notations, determinations and indexes here are identical to systems (1.1) and (1.2), except that they are in monetary terms.

Here as well as for I-O in physical terms, by means of system (2.1), the total required outputs of commodities are obtained for the given quantities of final uses in the certain conditions for the matrix of the direct input coefficients (A); (and) consequently, by the substitution of the obtained required output quantities in the system (2.2) the required quantity of primary factors are defined as v^d. If required quantities are less or equal to the available quantities ($v^d \le v_0$), then there is a quantitative equilibrium and the price equilibrium might be considered. Conversely, when at least the required quantity for one factor is larger than its available quantity, then the process must be carried out for the new different quantities for final uses, until the above condition will be satisfied.

The demand quantitative model system in monetary terms is widely used in practice, and it's worth while to consider the character of changes of the total required quantities of primary factors due to change of quantities of final uses similar to the demand quantitative model in physical terms (vide supra). To clarify the matter, let's assume that only the quantities of final use for a one of sector (commodity) l is changed (increased) (Δy_l^d), while final uses for other sectors (commodities) stay unchanged. Substitute this in (2.1) as we did in physical input-output (see (1.5), (1.6), (1.7)) and we have:

$$\Delta x_i^d = b_{ij}\Delta y_l^d,\ j = l,\ \left(i = 1, 2, \cdots, n\right) \tag{2.3}$$

From (2.3) we can conclude that increasing the final use of commodity of a certain sector either increases the total production of commodities of part of sectors when according inverse coefficients of inputs is more than zero or doesn't change if according inverse coefficients of inputs equal zero. Therefore, the quantities of primary factors are either increased if direct input coefficients of primary factors are more than zero ($\underline{c}_{kj} > 0$) or unchanged if direct input coefficients of primary factors are equal to zero ($\underline{c}_{kj} = 0$) in sectors where the total production is increased.

In addition, input-output in monetary terms in the equilibrium state is characterized by the balance between the total value added for all sectors and the total final uses for all sectors too ([18]; [5]):

$$\sum_{j=1}^{n} v_j^s = \sum_{i=1}^{n} y_i^d \tag{2.4}$$

This is also true for the particular case which is discussed. Changes (increasing) of value added in all sectors (all primary factors used in each sector) must be equal to the change in the final use of the sector in question, that is:

$$\sum_{j=1}^{n}\sum_{k=1}^{m} \Delta v_{kj}^s = \sum_{j=1}^{n} \Delta v_j^s = \Delta y_{i(=j)}^d \tag{2.5}$$

If we take into account (2.5), we can conclude that increasing the final use in the sector in question is generally more than increasing of value added in this sector.

By this we proved the following theorem:

Theorem 3 If matrix A is positive ($A \ge 0$) and productive ($x > xA$), and if quantities of final use of a certain sector y_l^d is increased and final uses for all other sectors are unchanged, then the quantities of primary factors are either increased if direct input coefficients are more than zero ($c_{kj} > 0$) or unchanged if they are equal zero ($c_{kj} = 0$) in sectors where the total production was increased; and the magnitude of the increase of final use of a certain sector (commodity) is limited by the unemployed supply quantities of primary factors; also the derived increase of value added of the sector in question is less than the increase of the final use in this sector

$$\Delta v_j^s = \Sigma_{k=1}^m \Delta v_{kj}^s \prec \Delta y_{i(=j)}^d \tag{2.6}$$

if at least one of $c_{kj} > 0$ when ($j \neq i$).

Similar to *Theorem* 1, this theorem indicates that increasing of quantities in the final use of commodity of a certain sector, increases required quantities of primary factors almost in all sectors; in addition, an increase of the total required quantities of primary factors for the sector in question is less than the increase of the final use in this sector.

On the basis of the above, we can also conclude that the supply quantitative equilibrium for I-O in money terms is identical to the supply quantitative equilibrium for I-O in physical terms and consists in the following systems:

$$X^{ss} = V^s \circ C^o, \tag{2.7}$$

$$x^{sj} = \min_{1 \le k \le m} \left\{ x^{skj} \right\}, \left(j = 1, 2, \cdots, n \right) \tag{2.8}$$

$$y^{ds} = \left(I - A \right) \left(x^s \right)' \tag{2.9}$$

where V^s, C^o, A—are given.

All notations and determinations here are identical to systems (1.29), (1.30) and (1.31), except that they are in monetary terms.

The Equation (2.7) defines matrix of possible total production of commodities for each primary factor by ordinary multiplication matrix of the flows of primary factors to branches (V^s) and matrix of direct output coefficients of primary factors (C^0). Here, there might be m different total quantity of commodity for a certain commodity. Consequently, the Equation system (2.8) allows us to choose one total quantity so that it might be possible from the point of all primary factors. Finally, the equation (2.9) allows us to obtain the final uses of commodities for choosing total quantities of production.

The character of changes of the total quantities of final uses for the supply quantitative model in monetary terms has additional economic sense because of the homogeneity of measurement of the monetary input-output. In this case, the value of different primary factors used for a certain branch and the value of different commodities demanded for a certain category of final uses might be summarized.

Because the supply quantitative equilibrium for I-O in money terms is identical to the supply quantitative equilibrium for I-O in physical terms we can conclude that the above considered properties (see *Theorem* 2) for the latter have to be correct also for the former in the same framework. We see that increasing the quantity of any primary factor for a certain sector increase the final use of this sector and decrease or don't change the final uses of all other sectors:

$$\Delta y_i^{ds} = \left(1 - a_{ij} \right) \Delta x_j^s, \text{ when } i = j \tag{2.10}$$

$$\begin{gathered} \Delta y_i^{ds} = \left(-a_{ij} \right) \Delta x_j^s, \\ \text{when } i \neq j, \left(i = 1, 2, \cdots, i = j-1, i = j+1, \cdots, n \right) \end{gathered} \tag{2.11}$$

These changes are derived from the changes (increases) of primary factors for branch j. Because of the balance between the total value added for all sectors and the total final uses for all sectors (2.4), changes (increasing) of value added in one sector (all primary factors used in this sector) must be equal to changes in final uses of all sectors, that is:

$$\sum_{k=1}^m \Delta v_{kj}^s = \Delta v_j^s = \Delta y = \sum_{i-1}^n \Delta y_i^{ds} = \Delta y_{i(=j)}^{ds} - \sum_{i=1}^n \Delta y_{i(\neq j)}^{ds} \tag{2.12}$$

If we take into account (2.12), we can conclude that increasing of final use in the sector in question deriving from the increasing of value added in this sector is generally more than the latter. By this we proved the following theorem:

Theorem 4 If matrix A is positive ($A \geq 0$) and productive ($x > xA$), and if quantities of all primary factors v_{kj}^s of a certain sector j is increased by the same rate, and primary factors for all other sectors are unchanged, then the final use of the sector in question y_i^{ds} ($i = j$) is increased and the final uses of other sectors y_i^{ds} ($I \neq j$) either decreased when $a_{ij} > 0$ or unchanged when $a_{ij} = 0$; and therefore, the derived increase of the final use of the sector in question is more than the increase of value added in this sector

$$\Delta v_j^s \prec \Delta y_i^{ds}, \text{ when } \left(i = j \right) \tag{2.13}$$

if at least one of $a_{ij} > 0$ ($i \neq j$); and the magnitude of the certain primary factor's (factors') increase in a certain sector is limited by the unsatisfied final uses of the sector in question and final uses of other sectors.

Here also, similar to *Theorem* 2, the increase of the quantity of any primary factor for a certain sector, increases the final use of this sector and decreases or doesn't change the final uses of all other sectors; and, in addition, derived increase of the final use in this sector in question is more than the increase of the total quantities of primary factors for the sector.

The properties of *Theorems* 3 and 4 might be illustrated by means of hypothetical input-output in monetary terms (for example $, which do not appear in the **Table 2**):

From the **Table 2** we can define the direct input coefficients of commodities (A) and primary factors (C) and consequently Leontief's inverse coefficients (B) and output coefficients of primary factors (C^o):

$$A = X\left(\hat{x}^d\right)^{-1} = \begin{pmatrix} 85.0 & 68.0 \\ 120.0 & 52.0 \end{pmatrix}\begin{pmatrix} 1/340.0 & 0 \\ 0 & 1/425.0 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.16 \\ 0.353 & 0.122 \end{pmatrix} \quad (2.14)$$

$$C = V\left(\hat{x}^d\right)^{-1} = \begin{pmatrix} 25.0 & 35.0 \\ 110.0 & 270.0 \end{pmatrix}\begin{pmatrix} 1/340.0 & 0 \\ 0 & 1/425.0 \end{pmatrix} = \begin{pmatrix} 0.074 & 0.082 \\ 0.323 & 0.636 \end{pmatrix} \quad (2.15)$$

$$B = (I - A)^{-1} = \begin{pmatrix} 1.458 & 0.266 \\ 0.586 & 1.246 \end{pmatrix} \quad (2.16)$$

and

$$C^o = 1/C = \begin{pmatrix} 13.5 & 12.2 \\ 3.1 & 1.6 \end{pmatrix} \quad (2.17).$$

Firstly, let us consider *Theorem* 3. For this purpose, let's assume that the final use of sector 2 is increased by 85.0 $ (Δy_2 = 85.0$); then the total production of sector 1 is increased by 22.6 $ ($\Delta x_1 = b_{12}\,\Delta y_2 = 0.266 \times 85.0 =$ 22.6$), and of sector 2 is increased by 105.9 $ ($\Delta x_2 = b_{22}\,\Delta y_2 = 1.246 \times 85.0 = 105.9$$). From this increasing of value added would be

$$\Delta V = \begin{pmatrix} 0.074 & 0.082 \\ 0.323 & 0.636 \end{pmatrix}\begin{pmatrix} 22.6 & 0 \\ 0 & 105.9 \end{pmatrix} = \begin{pmatrix} 1.65 & 8.66 \\ 7.3 & 67.35 \end{pmatrix} \quad (2.18)$$

And

$$\Delta v^s = \left(\Delta v_1^s \Delta v_2^s\right) = \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1.65 & 8.66 \\ 7.3 & 67.35 \end{pmatrix} = \begin{pmatrix} 8.95 & 76.01 \end{pmatrix} \quad (2.19)$$

So, Δv_2^s = 76.01$ and it is less than Δy_2 = 85.0$, what is according to the *Theorem* 3. It is worthy to stress that the total amount of increasing of value added in both sectors is equal to the amount of increasing of the final use of the second sector (= 85.0$).

Now, assume that the quantity of the first primary factor for the second sector is increased by 8.66$ (Δv_{12}^s = 8.66$); then the total output of the sector 2 is increased by ($\Delta x_2 = c_{12}^o \Delta v_{12}^s = 12.2 \times 8.66\$ = 105.7\$$), similar to the previous example (≅ 105.9). While the total output of the first sector is not changed. From this for the final uses we have

$$\Delta y^{ds} = \left(\Delta y_1^{ds} \Delta y_2^{ds}\right)' = (I - A)\Delta x^d = \begin{pmatrix} 0.75 & -0.16 \\ -0.353 & 0.878 \end{pmatrix}\begin{pmatrix} 0 \\ 105.7 \end{pmatrix} = \begin{pmatrix} -16.9 \\ 92.7 \end{pmatrix} \quad (2.20)$$

So, the final use of the second sector is increased by 92.7$, which is more than the total increase of the value added of this sector 75.8$, which is according to *Theorem* 4. At the same time the final use of the first sector is decreased by 16.9$. But, the total increase of final uses in both sectors (92.7$ – 16.9$ = 75.8$) is equal to the total increase of value added in the second sector. It is interesting to note that despite the fact that in both cases the total valued added of the second sector was increased in the same magnitude (75.8$), the final use of this sector increased by different magnitudes. This is due to the fact that in the second case only the total production of the second sector was changed (increased), while in the first case the total production of both sectors was changed (increased).

Therefore, in order to the final uses of the first sector is not changed it is necessary to increase its production in the magnitude which cover (equal) requirement to produce additional quantity of the second sector and the first sector itself. Namely, the input of the first primary factor

Table 2. Hypothetical input-output in monetary terms

	Agriculture	Manufacturing	Intermediate Total Output	Final Uses y^d	Total Output x^d
Agriculture	85.0	68.0	153.0	187.0	340.0
Manufacturing	120.0	52.0	172.0	253.0	425.0
Intermediate Total Input	205.0	120.0	325.0	440.0	765.0
Capital Stocks	25.0	35.0	60.0		
Labour	110.0	270.0	380.0		
Total value added v^s	135.0	305.0	440.0		
Total Input x^s	340.0	425.0			

of the first sector must be increased by 1.65$, or, alternatively, the input of the second primary factor of the first sector must be increased by 7.3$. Both cases the production of the first sector is increased by 22.6$ (13.5 × 1.65$ = 22.34, or 3.1 × 7.3$= 22.6$).

To complete the demonstration of the above mentioned statement that the supply quantitative model à la Leontief is equivalent to the demand quantitative model of Leontief assume that the value added of both sectors are increased by 8.95$ and 76.0$ respectively. Then, the quantity of the first sector is increased by 22.6$ (2.52 × 8.95$ = 22.6$) and the quantity of the second sector is increased by 105.8$ (1.393 × 76.0$= 105.8$). Finally, the final uses of sectors will be increased

$$\begin{pmatrix} 0.75 & -0.16 \\ -0.353 & 0.878 \end{pmatrix}\begin{pmatrix} 22.6 \\ 105.7 \end{pmatrix}=\begin{pmatrix} 0 \\ 85.0 \end{pmatrix}.$$

which are identical with the results of the demand quantitative model (vide supra).

4. Conclusions

This paper examines the supply quantitative model system of input–output for both physical and monetary terms, which is equivalent to the demand quantitative model system of Leontief. The supply quantitative model is based on the direct output coefficients of primary factors which are the inversion of the direct inputs coefficients.

This paper also took into consideration the properties of both demand and supply quantitative system models. It was shown that: (1) the increase of the final use of a certain sector generally increases required (demand) quantities of primary factors for all sectors and magnitude of the increase of final use of a certain sector (commodity) is limited by the unemployed supply quantities of primary factors in the quantitative demand model system in physical terms (see *Theorem* 1); (2) the increase of the primary factors of a certain sector increases final use of this sector and generally decreases final uses of the rest sectors and the magnitude of the increase of a certain primary factor (factors) in a certain sector is limited by the unsatisfied final uses of the sector in question and by the final uses of other sectors in the quantitative supply model system in physical terms (see *Theorem* 2); (3) in the quantitative demand model in monetary terms the increase of the total value added of a certain sector, deriving from the increasing quantity of final use of the sector in question, is generally less than the latter and magnitude of the increase of final use of a certain sector (commodity) is limited by the unemployed supply quantities of primary factors (see *Theorem* 3); and (4) in the quantitative supply model system in monetary terms the increase of final use of a certain sector, deriving from the increasing of the value added of the sector in question, is greater than the increase of the total value added of this sector and the magnitude of the increase of a certain primary factor (factors) in a certain sector is limited by the unsatisfied final uses of the sector in question and by the final uses of other sectors (see *Theorem* 4).

Finally, the quantitative supply system models might be useful tools in planning the economics of countries that have higher unemployment of primary factors, especially labour.

5. References

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