

Article

New Hadamard and Fejér-Hadamard fractional inequalities for exponentially m -convex function

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Abstract: In this article, we present new fractional Hadamard and Fejér-Hadamard inequalities for generalized fractional integral operators containing Mittag-Leffler function via a monotone function. To establish these inequalities we will use exponentially m -convex functions. The presented results in particular contain a number of fractional Hadamard and Fejér-Hadamard inequalities for functions deducible from exponentially m -convex functions.

Keywords: Convex functions, exponentially m -convex functions, Hadamard inequality, Fejér-Hadamard inequality, generalized fractional integral operators, Mittag-Leffler function.

1. Introduction and Preliminaries

A real valued function $\eta : I \rightarrow \mathbb{R}$ is said to be convex on I , if the following inequality holds:

$$\eta(\tau\zeta_1 + (1 - \tau)\zeta_2) \leq \tau\eta(\zeta_1) + (1 - \tau)\eta(\zeta_2), \quad \forall \zeta_1, \zeta_2 \in I, \quad \tau \in [0, 1]. \quad (1)$$

The function η is said to be concave if reversed of inequality (1) holds.

A convex function is also equally defined by the well known Hadamard inequality stated as follows:

$$\eta\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \eta(\tau) d\tau \leq \frac{\eta(\zeta_1) + \eta(\zeta_2)}{2},$$

where $\eta : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ is a convex function.

In [1], Fejér gave the generalization of Hadamard inequality stated as follows:

$$\eta\left(\frac{\zeta_1 + \zeta_2}{2}\right) \int_{\zeta_1}^{\zeta_2} \kappa(\tau) d\tau \leq \int_{\zeta_1}^{\zeta_2} \eta(\tau) \kappa(\tau) d\tau \leq \frac{\eta(\zeta_1) + \eta(\zeta_2)}{2} \int_{\zeta_1}^{\zeta_2} \kappa(\tau) d\tau, \quad (2)$$

where $\eta : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ is convex function and $\kappa : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}$ is a positive, integrable and symmetric to $\frac{\zeta_1 + \zeta_2}{2}$.

The inequality (2) is well known as the Fejér-Hadamard inequality. The Hadamard and the Fejér-Hadamard inequalities have been analyzed by many researchers and produced frequently their generalizations, refinements and extensions (see, [2–18]).

In [19], Rashid *et al.*, introduced the concept of exponentially m -convex functions defined as follows:

Definition 1. A real-valued function $\eta : I \rightarrow \mathbb{R}$ is said to be exponentially m -convex, if the following inequality holds:

$$e^{\eta(\tau\zeta_1 + m(1-\tau)\zeta_2)} \leq \tau e^{\eta(\zeta_1)} + m(1 - \tau)e^{\eta(\zeta_2)}, \quad \forall \zeta_1, \zeta_2 \in I, \quad m \in (0, 1], \quad \tau \in [0, 1]. \quad (3)$$

If we take $m = 1$ in (3), then exponentially convex function defined by Antczak in [20] is obtained, see also [6]. We recall that a real-valued function $\eta : I \rightarrow \mathbb{R}$ is said to be exponentially convex, if the following inequality holds:

$$e^{\eta(\tau\zeta_1 + (1-\tau)\zeta_2)} \leq \tau e^{\eta(\zeta_1)} + (1 - \tau)e^{\eta(\zeta_2)}, \quad \forall \zeta_1, \zeta_2 \in I, \quad \tau \in [0, 1]. \quad (4)$$

Next we give the definition of generalized fractional integral operators containing Mittag-Leffler function in their kernels as follows:

Definition 2. [21] Let $\omega, \vartheta, \theta, l, \rho, c \in \mathbb{C}, \Re(\vartheta), \Re(\theta), \Re(l) > 0, \Re(c) > \Re(\rho) > 0$ with $p \geq 0, r > 0$ and $0 < q \leq r + \Re(\vartheta)$. Let $\eta \in L_1[\zeta_1, \zeta_2]$ and $u \in [\zeta_1, \zeta_2]$. Then the generalized fractional integral operators $Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} \eta$ and $Y_{\vartheta, \theta, l, \omega, \zeta_2^-}^{\rho, r, q, c} \eta$ are defined by:

$$\left(Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} \eta \right) (u; p) = \int_{\zeta_1}^u (u - \tau)^{\theta-1} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(u - \tau)^\vartheta; p) \eta(\tau) d\tau, \tag{5}$$

$$\left(Y_{\vartheta, \theta, l, \omega, \zeta_2^-}^{\rho, r, q, c} \eta \right) (u; p) = \int_u^{\zeta_2} (\tau - u)^{\theta-1} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(\tau - u)^\vartheta; p) \eta(\tau) d\tau, \tag{6}$$

where $E_{\vartheta, \theta, l}^{\rho, r, q, c}(\tau; p)$ is the generalized Mittag-Leffler function defined as follows:

$$E_{\vartheta, \theta, l}^{\rho, r, q, c}(\tau; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nq, c - \rho)}{\beta(\rho, c - \rho)} \frac{(c)_{nq}}{\Gamma(\vartheta n + \theta)} \frac{\tau^n}{(l)_{nr}}.$$

In [22], Farid defined the following unified integral operators:

Definition 3. Let $\eta, \kappa : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}, 0 < \zeta_1 < \zeta_2$ be the functions such that η be a positive and integrable and κ be a differentiable and strictly increasing. Also, let $\frac{\gamma}{u}$ be an increasing function on $[\zeta_1, \infty)$ and $\theta, l, \rho, c \in \mathbb{C}, p, \vartheta, r \geq 0$ and $0 < q \leq r + \vartheta$. Then for $u \in [\zeta_1, \zeta_2]$ the integral operators ${}_{\kappa} Y_{\vartheta, \theta, l, \zeta_1^+}^{\gamma, \rho, r, q, c} \eta$ and ${}_{\kappa} Y_{\vartheta, \theta, l, \zeta_2^-}^{\gamma, \rho, r, q, c} \eta$ are defined by:

$$\left({}_{\kappa} Y_{\vartheta, \theta, l, \zeta_1^+}^{\gamma, \rho, r, q, c} \eta \right) (u; p) = \int_{\zeta_1}^u \frac{\gamma(\kappa(u) - \kappa(\tau))}{\kappa(u) - \kappa(\tau)} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(\kappa(u) - \kappa(\tau))^\vartheta; p) \eta(\tau) d(\kappa(\tau)), \tag{7}$$

$$\left({}_{\kappa} Y_{\vartheta, \theta, l, \zeta_2^-}^{\gamma, \rho, r, q, c} \eta \right) (u; p) = \int_u^{\zeta_2} \frac{\gamma(\kappa(\tau) - \kappa(u))}{\kappa(\tau) - \kappa(u)} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(\kappa(\tau) - \kappa(u))^\vartheta; p) \eta(\tau) d(\kappa(\tau)). \tag{8}$$

If we take $\gamma(u) = u^\theta$ in (7) and (8), then we get the following generalized fractional integral operators containing Mittag-Leffler function:

Definition 4. Let $\eta, \kappa : [\zeta_1, \zeta_2] \rightarrow \mathbb{R}, 0 < \zeta_1 < \zeta_2$ be the functions such that η be a positive and integrable and κ be a differentiable and strictly increasing. Also, let $\vartheta, \theta, l, \omega, \rho, c \in \mathbb{C}, p, \vartheta, r \geq 0$ and $0 < q \leq r + \vartheta$. Then for $u \in [\zeta_1, \zeta_2]$ the integral operators ${}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} \eta$ and ${}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_2^-}^{\rho, r, q, c} \eta$ are defined by:

$$\left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} \eta \right) (u; p) = \int_{\zeta_1}^u (\kappa(u) - \kappa(\tau))^{\theta-1} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(\kappa(u) - \kappa(\tau))^\vartheta; p) \eta(\tau) d(\kappa(\tau)), \tag{9}$$

$$\left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_2^-}^{\rho, r, q, c} \eta \right) (u; p) = \int_u^{\zeta_2} (\kappa(\tau) - \kappa(u))^{\theta-1} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(\kappa(\tau) - \kappa(u))^\vartheta; p) \eta(\tau) d(\kappa(\tau)). \tag{10}$$

Remark 1. (9) and (10) are the generalization of the following fractional integral operators:

1. By taking $\kappa(u) = u$, the fractional integral operators (5) and (6), can be achieved.
2. By taking $\kappa(u) = u$ and $p = 0$, the fractional integral operators defined by Salim-Faraj in [23], can be achieved.
3. By taking $\kappa(u) = u$ and $l = r = 1$, the fractional integral operators defined by Rahman *et al.*, in [24], can be achieved.
4. By taking $\kappa(u) = u, p = 0$ and $l = r = 1$, the fractional integral operators defined by Srivastava-Tomovski in [25], can be achieved.
5. By taking $\kappa(u) = u, p = 0$ and $l = r = q = 1$, the fractional integral operators defined by Prabhakar in [26], can be achieved.

6. By taking $\kappa(u) = u$ and $\omega = p = 0$, the Riemann-Liouville fractional integral operators can be achieved.

From generalized fractional integral operator (9), we have

$$\begin{aligned} \left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} 1 \right) (u; p) &= \int_{\zeta_1}^u (\kappa(u) - \kappa(\tau))^{\theta-1} E_{\vartheta, \theta, l}^{\rho, r, q, c} (\omega(\kappa(u) - \kappa(\tau))^{\vartheta}; p) d(\kappa(\tau)) \\ &= \int_{\zeta_1}^u (\kappa(u) - \kappa(\tau))^{\theta-1} \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nq, c - \rho)}{\beta(\rho, c - \rho)} \frac{(c)_{nq}}{\Gamma(\vartheta n + \theta)} \frac{\omega^n (\kappa(u) - \kappa(\tau))^{\vartheta n}}{(l)_{nr}} d(\kappa(\tau)) \\ &= \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nq, c - \rho)}{\beta(\rho, c - \rho)} \frac{(c)_{nq}}{\Gamma(\vartheta n + \theta)} \frac{\omega^n}{(l)_{nr}} \int_{\zeta_1}^u (\kappa(u) - \kappa(\tau))^{\vartheta n + \theta - 1} d(\kappa(\tau)) \\ &= (\kappa(u) - \kappa(\zeta_1))^{\theta} \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nq, c - \rho)}{\beta(\rho, c - \rho)} \frac{(c)_{nq}}{\Gamma(\vartheta n + \theta)} \frac{\omega^n}{(l)_{nr}} \frac{(\kappa(u) - \kappa(\zeta_1))^{\vartheta n}}{\vartheta n + \theta} \\ &= (\kappa(u) - \kappa(\zeta_1))^{\theta} \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nq, c - \rho)}{\beta(\rho, c - \rho)} \frac{(c)_{nq}}{\Gamma(\vartheta n + \theta + 1)} \frac{\omega^n (\kappa(u) - \kappa(\zeta_1))^{\vartheta n}}{(l)_{nr}} \\ &= (\kappa(u) - \kappa(\zeta_1))^{\theta} E_{\vartheta, \theta + 1, l}^{\rho, r, q, c} (\omega(\kappa(u) - \kappa(\zeta_1))^{\vartheta}; p). \end{aligned}$$

Hence

$$\left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} 1 \right) (u; p) = (\kappa(u) - \kappa(\zeta_1))^{\theta} E_{\vartheta, \theta + 1, l}^{\rho, r, q, c} (\omega(\kappa(u) - \kappa(\zeta_1))^{\vartheta}; p)$$

and similarly, from generalized fractional integral operator (10), we get

$$\left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_2^-}^{\rho, r, q, c} 1 \right) (u; p) = (\kappa(\zeta_2) - \kappa(u))^{\theta} E_{\vartheta, \theta + 1, l}^{\rho, r, q, c} (\omega(\kappa(\zeta_2) - \kappa(u))^{\vartheta}; p).$$

We will use the following notations in the article:

$${}_{\kappa} \zeta_{\omega, \zeta_1^+}^{\theta} (u; p) = \left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_1^+}^{\rho, r, q, c} 1 \right) (u; p), \tag{11}$$

$${}_{\kappa} \zeta_{\omega, \zeta_2^-}^{\theta} (u; p) = \left({}_{\kappa} Y_{\vartheta, \theta, l, \omega, \zeta_2^-}^{\rho, r, q, c} 1 \right) (u; p). \tag{12}$$

In [27], Mehmood *et al.*, proved the following Hadamard and Fejér-Hadamard inequalities for exponentially m -convex functions via generalized fractional integral operators (5) and (6).

Theorem 5. Let $\eta : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\eta \in L_1[\zeta_1, m\zeta_2]$ with $\zeta_1 < m\zeta_2$. If η is exponentially m -convex function, then the following inequalities hold:

$$\begin{aligned} e^{\eta\left(\frac{\zeta_1 + m\zeta_2}{2}\right)} \zeta_{\theta, \bar{\omega}, \zeta_1^+} (m\zeta_2; p) &\leq \frac{\left(Y_{\vartheta, \theta, l, \bar{\omega}, \zeta_1^+}^{\rho, r, q, c} e^{\eta} \right) (m\zeta_2; p) + m^{\theta+1} \left(Y_{\vartheta, \theta, l, \bar{\omega} m^{\vartheta}, \zeta_2^-}^{\rho, r, q, c} e^{\eta} \right) \left(\frac{\zeta_1}{m}; p \right)}{2} \\ &\leq \frac{m^{\theta+1}}{2(m\zeta_2 - \zeta_1)} \left[\left(e^{\eta(\zeta_1)} - m^2 e^{\eta\left(\frac{\zeta_1}{m^2}\right)} \right) \zeta_{\theta+1, \bar{\omega} m^{\vartheta}, \zeta_2^-} \left(\frac{\zeta_1}{m}; p \right) + (m\zeta_2 - \zeta_1) \left(e^{\eta(\zeta_2)} + m e^{\eta\left(\frac{\zeta_1}{m^2}\right)} \right) \zeta_{\theta, \bar{\omega} m^{\vartheta}, \zeta_2^-} \left(\frac{\zeta_1}{m}; p \right) \right], \end{aligned}$$

where $\bar{\omega} = \frac{\omega}{(m\zeta_2 - \zeta_1)^{\vartheta}}$.

Theorem 6. Let $\eta : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\eta \in L_1[\zeta_1, m\zeta_2]$ with $\zeta_1 < m\zeta_2$. If η is exponentially m -convex function, then the following inequalities hold:

$$\begin{aligned} e^{\eta\left(\frac{\zeta_1 + m\zeta_2}{2}\right)} \zeta_{\theta, \bar{\omega} 2^{\vartheta}, \left(\frac{\zeta_1 + m\zeta_2}{2}\right)^+} (m\zeta_2; p) &\leq \frac{\left(Y_{\vartheta, \theta, l, \bar{\omega} 2^{\vartheta}, \left(\frac{\zeta_1 + m\zeta_2}{2}\right)^+}^{\rho, r, q, c} e^{\eta} \right) (m\zeta_2; p) + m^{\theta+1} \left(Y_{\vartheta, \theta, l, \bar{\omega} (2m)^{\vartheta}, \left(\frac{\zeta_1 + m\zeta_2}{2m}\right)^-} e^{\eta} \right) \left(\frac{\zeta_1}{m}; p \right)}{2} \\ &\leq \frac{m^{\theta+1}}{2(m\zeta_2 - \zeta_1)} \left[\left(e^{\eta(\zeta_1)} - m^2 e^{\eta\left(\frac{\zeta_1}{m^2}\right)} \right) \zeta_{\theta+1, \bar{\omega} (2m)^{\vartheta}, \left(\frac{\zeta_1 + m\zeta_2}{2m}\right)^-} \left(\frac{\zeta_1}{m}; p \right) + (m\zeta_2 - \zeta_1) \left(e^{\eta(\zeta_2)} + m e^{\eta\left(\frac{\zeta_1}{m^2}\right)} \right) \right] \end{aligned}$$

$$\times \zeta_{\theta, \bar{\omega}(2m)^\theta, \left(\frac{\zeta_1+m\zeta_2}{2m}\right)^- \left(\frac{\zeta_1}{m}; p\right)} \Big],$$

where $\bar{\omega} = \frac{\omega}{(m\zeta_2 - \zeta_1)^\theta}$.

Theorem 7. Let $\eta : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $\eta \in L_1[\zeta_1, m\zeta_2]$ with $\zeta_1 < m\zeta_2$. Also, let $\kappa : [\zeta_1, m\zeta_2] \rightarrow \mathbb{R}$ be a function which is non-negative and integrable. If η is exponentially m -convex function and $\eta(v) = \eta(\zeta_1 + m\zeta_2 - mv)$, then the following inequalities hold:

$$\begin{aligned} e^{\eta\left(\frac{\zeta_1+m\zeta_2}{2}\right)} \left(Y_{\theta, \theta, l, \bar{\omega}m^\theta, \zeta_2^-}^{\rho, r, q, c} e^\kappa \right) \left(\frac{\zeta_1}{m}; p \right) &\leq \frac{(1+m) \left(Y_{\theta, \theta, l, \bar{\omega}m^\theta, \zeta_2^-}^{\rho, r, q, c} e^\eta e^\kappa \right) \left(\frac{\zeta_1}{m}; p \right)}{2} \\ &\leq \frac{m}{2(m\zeta_2 - \zeta_1)} \left[\left(e^{\eta(\zeta_1)} - m^2 e^{\eta\left(\frac{\zeta_1}{m^2}\right)} \right) \left(Y_{\theta, \theta+1, l, \bar{\omega}m^\theta, \zeta_2^-}^{\rho, r, q, c} e^\kappa \right) \left(\frac{\zeta_1}{m}; p \right) + (m\zeta_2 - \zeta_1) \left(e^{\eta(\zeta_2)} + m e^{\eta\left(\frac{\zeta_1}{m^2}\right)} \right) \right. \\ &\quad \left. \times \left(Y_{\theta, \theta, l, \bar{\omega}m^\theta, \zeta_2^-}^{\rho, r, q, c} e^\kappa \right) \left(\frac{\zeta_1}{m}; p \right) \right], \end{aligned}$$

where $\bar{\omega} = \frac{\omega}{(m\zeta_2 - \zeta_1)^\theta}$.

In this article, we establish the Hadamard and the Fejér-Hadamard inequalities for exponentially m -convex functions by the generalized fractional integral operators (9) and (10) containing Mittag-Leffler function via a monotone function. These inequalities lead to produce results for generalized fractional integral operators given in Remark 1. In Section 2, we prove the Hadamard inequalities for generalized fractional integral operators (9) and (10) via exponentially m -convex functions. In Section 3, we prove the Fejér-Hadamard inequalities for these generalized fractional integral operators via exponentially m -convex functions. In whole paper, we will consider real parameters of the Mittag-Leffler function.

2. Hadamard inequalities for exponentially m -convex functions

Here we will give two versions of the generalized fractional Hadamard inequality.

Theorem 8. Let $\eta, \kappa : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}$, $0 < \zeta_1 < m\zeta_2$ be the real valued-functions. If η be a integrable and exponentially m -convex and κ be a differentiable and strictly increasing. Then for integral operators (9) and (10), the following inequalities hold:

$$\begin{aligned} &2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \kappa \zeta_{\bar{\omega}, \zeta_1^+}^\theta (\kappa^{-1}(m\kappa(\zeta_2)); p) \\ &\leq \left(\kappa Y_{\theta, \theta, l, \bar{\omega}, \zeta_1^+}^{\rho, r, q, c} e^{\eta \circ \kappa} \right) (\kappa^{-1}(m\kappa(\zeta_2)); p) r + m^{\theta+1} \left(\kappa Y_{\theta, \theta, l, \bar{\omega}m^\theta, \zeta_2^-}^{\rho, r, q, c} e^{\eta \circ \kappa} \right) \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \\ &\leq \frac{m^{\theta+1}}{(m\kappa(\zeta_2) - \kappa(\zeta_1))} \left[\left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \zeta_{\bar{\omega}m^\theta, \zeta_2^-}^{\theta+1} \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \right. \\ &\quad \left. + \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) (m\kappa(\zeta_2) - \kappa(\zeta_1)) \kappa \zeta_{\bar{\omega}m^\theta, \zeta_2^-}^\theta \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \right] \end{aligned} \tag{13}$$

where $\bar{\omega} = \frac{\omega}{(m\kappa(\zeta_2) - \kappa(\zeta_1))^\theta}$.

Proof. Since η is exponentially m -convex function on $[\zeta_1, m\zeta_2]$, for $\tau \in [0, 1]$, we have

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \leq e^{\eta(\tau\kappa(\zeta_1)+m(1-\tau)\kappa(\zeta_2))} + m e^{\eta\left((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2)\right)}. \tag{14}$$

Also, from exponentially m -convexity, we have

$$e^{\eta(\tau\kappa(\zeta_1)+m(1-\tau)\kappa(\zeta_2))} + m e^{\eta\left((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2)\right)} \leq \tau \left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) + m \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right). \tag{15}$$

Multiplying both sides of (14) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p)$ and integrating over $[0, 1]$, we have

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) d\tau \leq \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta(\tau\kappa(\zeta_1)+m(1-\tau)\kappa(\zeta_2))} d\tau + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta\left((1-\tau)\frac{\kappa(\zeta_1)}{m}+\tau\kappa(\zeta_2)\right)} d\tau. \tag{16}$$

Putting $\kappa(u) = \tau\kappa(\zeta_1) + m(1 - \tau)\kappa(\zeta_2)$ and $\kappa(v) = (1 - \tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2)$ in (16), we get

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \int_{\zeta_1}^{\kappa^{-1}(m\kappa(\zeta_2))} (m\kappa(\zeta_2) - \kappa(u))^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\bar{\omega}(m\kappa(\zeta_2) - \kappa(u))^\theta; p) d(\kappa(u)) \leq \int_{\zeta_1}^{\kappa^{-1}(m\kappa(\zeta_2))} (m\kappa(\zeta_2) - \kappa(u))^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\bar{\omega}(m\kappa(\zeta_2) - \kappa(u))^\theta; p) e^{\eta(\kappa(u))} d(\kappa(u)) + m^{\theta+1} \int_{\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right)}^{\zeta_2} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m}\right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}\left(m^\theta \bar{\omega} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m}\right)^\theta; p\right) e^{\eta(\kappa(v))} d(\kappa(v)).$$

By using (9), (10) and (11), the first inequality of (13) is obtained.

Now multiplying both sides of (15) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p)$ and integrating over $[0, 1]$, we have

$$\int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta(\tau\kappa(\zeta_1)+m(1-\tau)\kappa(\zeta_2))} d\tau + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta\left((1-\tau)\frac{\kappa(\zeta_1)}{m}+\tau\kappa(\zeta_2)\right)} d\tau \leq \left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)}\right) \int_0^1 \tau^\theta E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) d\tau + m \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)}\right) \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) d\tau. \tag{17}$$

Putting $\kappa(u) = \tau\kappa(\zeta_1) + m(1 - \tau)\kappa(\zeta_2)$ and $\kappa(v) = (1 - \tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2)$ in (17), then by using (9), (10) and (12), the second inequality of (13) is obtained. □

Corollary 1. Under the assumptions of Theorem 8 if we take $m = 1$, then we get following inequalities for exponentially convex function:

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+\kappa(\zeta_2)}{2}\right)} \kappa \zeta_{\bar{\omega}, \zeta_1^+}^\theta(\zeta_2; p) \leq \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega}, \zeta_1^+}^{\rho, r, q, c} e^{\eta \circ \kappa}\right)(\zeta_2; p) + \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega}, \zeta_2^-}^{\rho, r, q, c} e^{\eta \circ \kappa}\right)(\zeta_1; p) \leq \left(e^{\eta(\kappa(\zeta_2))} + e^{\eta(\kappa(\zeta_1))}\right) \kappa \zeta_{\bar{\omega}, \zeta_2^-}^\theta(\zeta_1; p) \tag{18}$$

where $\bar{\omega} = \frac{\omega}{(\kappa(\zeta_2) - \kappa(\zeta_1))^\theta}$.

- Remark 2.**
1. If we take $\kappa(u) = u$ in (13), then Theorem 5 is obtained.
 2. If we take $\kappa(u) = u$ and $m = 1$ in (13), then [15, Corollary 2.2] is obtained.
 3. If we take $\kappa(u) = u$ in (18), then [15, Corollary 2.2] is obtained.

In the following we give another version of the Hadamard inequality for generalized fractional integral operators.

Theorem 9. Let $\eta, \kappa : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}, 0 < \zeta_1 < m\zeta_2$ be the real-valued functions. If η be a integrable and exponentially m -convex and κ be a differentiable and strictly increasing. Then for integral operators (9) and (10), the following inequalities hold:

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \kappa \zeta_{\bar{\omega} 2^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)\right)^+}^\theta + (\kappa^{-1}(m\kappa(\zeta_2)); p) \leq \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega} 2^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)\right)^+}^{\rho, r, q, c} e^{\eta \circ \kappa}\right)(\kappa^{-1}(m\kappa(\zeta_2)); p) + m^{\theta+1} \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega} (2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2m}\right)\right)^-}^{\rho, r, q, c} e^{\eta \circ \kappa}\right)\left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p\right)$$

$$\leq \frac{m^{\theta+1}}{(m\kappa(\zeta_2) - \kappa(\zeta_1))} \left[\left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \zeta_{\bar{\omega}(2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2m}\right)\right)}^{\theta+1} - \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p \right) \right. \\ \left. + \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) (m\kappa(\zeta_2) - \kappa(\zeta_1)) \kappa \zeta_{\bar{\omega}(2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2m}\right)\right)}^\theta - \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p \right) \right], \tag{19}$$

where $\bar{\omega} = \frac{\omega}{(m\kappa(\zeta_2) - \kappa(\zeta_1))^\theta}$.

Proof. Since η is exponentially m -convex function on $[\zeta_1, m\zeta_2]$, for $\tau \in [0, 1]$, we have

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \leq e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_1)+m\frac{(2-\tau)}{2}\kappa(\zeta_2)\right)} + me^{\eta\left(\frac{\tau}{2}\kappa(\zeta_2)+\frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)}. \tag{20}$$

Also, from exponentially m -convexity, we have

$$e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_1)+m\frac{(2-\tau)}{2}\kappa(\zeta_2)\right)} + me^{\eta\left(\frac{\tau}{2}\kappa(\zeta_2)+\frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)} \leq \frac{\tau}{2} \left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) + m \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right). \tag{21}$$

Multiplying both sides of (20) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p)$ and integrating over $[0, 1]$, we have

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) d\tau \\ \leq \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_1)+m\frac{(2-\tau)}{2}\kappa(\zeta_2)\right)} d\tau + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_2)+\frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)} d\tau. \tag{22}$$

Putting $\kappa(u) = \frac{\tau}{2}\kappa(\zeta_1) + m\frac{(2-\tau)}{2}\kappa(\zeta_2)$ and $\kappa(v) = \frac{\tau}{2}\kappa(\zeta_2) + \frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}$ in (22), we get

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)} \int_{\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)}^{\kappa^{-1}(m\kappa(\zeta_2))} (m\kappa(\zeta_2) - \kappa(u))^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(2^\theta \bar{\omega}(m\kappa(\zeta_2) - \kappa(u))^\theta; p) d(\kappa(u)) \\ \leq \int_{\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2}\right)}^{\kappa^{-1}(m\kappa(\zeta_2))} (m\kappa(\zeta_2) - \kappa(u))^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(2^\theta \bar{\omega}(m\kappa(\zeta_2) - \kappa(u))^\theta; p) e^{\eta(\kappa(u))} d(\kappa(u)) \\ + m^{\theta+1} \int_{\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right)}^{\kappa^{-1}\left(\frac{\kappa(\zeta_1)+m\kappa(\zeta_2)}{2m}\right)} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}\left((2m)^\theta \bar{\omega}(\kappa(v) - \frac{\kappa(\zeta_1)}{m})^\theta; p \right) e^{\eta(\kappa(v))} d(\kappa(v)).$$

By using (9), (10) and (11), the first inequality of (19) is obtained.

Now multiplying both sides of (21) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p)$ and integrating over $[0, 1]$, we have

$$\int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_1)+m\frac{(2-\tau)}{2}\kappa(\zeta_2)\right)} d\tau + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_2)+\frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)} d\tau \\ \leq \frac{1}{2} \left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \int_0^1 \tau^\theta E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) d\tau + m \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega\tau^\theta; p) d\tau. \tag{23}$$

Putting $\kappa(u) = \frac{\tau}{2}\kappa(\zeta_1) + m\frac{(2-\tau)}{2}\kappa(\zeta_2)$ and $\kappa(v) = \frac{\tau}{2}\kappa(\zeta_2) + \frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}$ in (23), then by using (9), (10) and (12), the second inequality of (19) is obtained. \square

Corollary 2. Under the assumptions of Theorem 9 if we take $m = 1$, then we get following inequalities for exponentially convex function:

$$2e^{\eta\left(\frac{\kappa(\zeta_1)+\kappa(\zeta_2)}{2}\right)} \kappa \zeta_{\bar{\omega}2^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+\kappa(\zeta_2)}{2}\right)\right)}^\theta + (\zeta_2; p) \leq \left(\kappa Y_{\theta, \theta, l, \bar{\omega}2^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)+\kappa(\zeta_2)}{2}\right)\right)}^{\rho, r, q, c} + e^{\eta \circ \kappa} \right) (\zeta_2; p)$$

$$+ \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} 2^{\theta}}^{\rho, r, q, c} \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2} \right) \right) e^{\eta \circ \kappa} \right) (\zeta_1; p) \leq \left(e^{\eta(\kappa(\zeta_2))} + e^{\eta(\kappa(\zeta_1))} \right) {}_{\kappa} \zeta_{\bar{\omega} 2^{\theta}}^{\theta} \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2} \right) \right) (\zeta_1; p), \quad (24)$$

where $\bar{\omega} = \frac{\omega}{(\kappa(\zeta_2) - \kappa(\zeta_1))^{\theta}}$.

- Remark 3.**
1. If we take $\kappa(u) = u$ in (19), then Theorem 6 is obtained.
 2. If we take $\kappa(u) = u$ and $m = 1$ in (19), then [15, Corollary 2.5] is obtained.
 3. If we take $\kappa(u) = u$ in (24), then [15, Corollary 2.5] is obtained.

3. Fejér-Hadamard Inequalities for exponentially m -convex functions

Here we give two versions of the Fejér-Hadamard inequality.

Theorem 10. Let $\eta, \kappa : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}$, $0 < \zeta_1 < m\zeta_2$ be the real-valued functions. If η be a integrable, exponentially m -convex and $\eta(\kappa(v)) = \eta(\kappa(\zeta_1)) + m\kappa(\zeta_2) - m\kappa(v)$ and κ be a differentiable and strictly increasing. Also, let $\gamma : [\zeta_1, m\zeta_2] \rightarrow \mathbb{R}$ be a function which is non-negative and integrable. Then for integral operators (9) and (10), the following inequalities hold:

$$\begin{aligned} & 2e^{\eta \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2} \right)} \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} m^{\theta}, \zeta_2}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \\ & \leq (1 + m) \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} m^{\theta}, \zeta_2}^{\rho, r, q, c} e^{\eta \circ \kappa} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \\ & \leq \frac{m}{(m\kappa(\zeta_2) - \kappa(\zeta_1))} \left[\left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta \left(\frac{\kappa(\zeta_1)}{m^2} \right)} \right) \left({}_{\kappa} Y_{\theta, \theta+1, l, \bar{\omega} m^{\theta}, \zeta_2}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \right. \\ & \quad \left. + \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta \left(\frac{\kappa(\zeta_1)}{m^2} \right)} \right) (m\kappa(\zeta_2) - \kappa(\zeta_1)) \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} m^{\theta}, \zeta_2}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right); p \right) \right], \quad (25) \end{aligned}$$

where $\bar{\omega} = \frac{\omega}{(m\kappa(\zeta_2) - \kappa(\zeta_1))^{\theta}}$.

Proof. Multiplying both sides of (14) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^{\theta}; p) e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2))}$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & 2e^{\eta \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2} \right)} \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^{\theta}; p) e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2))} d\tau \\ & \leq \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^{\theta}; p) e^{\eta(\tau\kappa(\zeta_1) + m(1-\tau)\kappa(\zeta_2))} e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2))} d\tau \\ & \quad + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^{\theta}; p) e^{\eta((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2))} e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2))} d\tau. \quad (26) \end{aligned}$$

Putting $\kappa(v) = (1 - \tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2)$ in (26), we get

$$\begin{aligned} & 2e^{\eta \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2} \right)} \int_{\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right)}^{\zeta_2} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c} \left(m^{\theta} \bar{\omega} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta}; p \right) e^{\gamma(\kappa(v))} d(\kappa(v)) \\ & \leq \int_{\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right)}^{\zeta_2} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c} \left(m^{\theta} \bar{\omega} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta}; p \right) e^{\eta(\kappa(\zeta_1) + m\kappa(\zeta_2) - m\kappa(v))} e^{\gamma(\kappa(v))} d(\kappa(v)) \\ & \quad + m \int_{\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right)}^{\zeta_2} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c} \left(m^{\theta} \bar{\omega} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta}; p \right) e^{\eta(\kappa(v))} e^{\gamma(\kappa(v))} d(\kappa(v)). \end{aligned}$$

By using (10) and given condition $\eta(\kappa(v)) = \eta(\kappa(\zeta_1)) + m\kappa(\zeta_2) - m\kappa(v)$, the first inequality of (25) is obtained.

Now multiplying both sides of (15) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^{\theta}; p) e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau\kappa(\zeta_2))}$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta(\tau \kappa(\zeta_1) + m(1-\tau)\kappa(\zeta_2))} e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau \kappa(\zeta_2))} d\tau \\ & + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau \kappa(\zeta_2))} e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau \kappa(\zeta_2))} d\tau. \\ & \leq \left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \int_0^1 \tau^\theta E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau \kappa(\zeta_2))} d\tau \\ & + m \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma((1-\tau)\frac{\kappa(\zeta_1)}{m} + \tau \kappa(\zeta_2))} d\tau. \end{aligned}$$

From above the second inequality of (25) is achieved. \square

Corollary 3. Under the assumptions of Theorem 10 if we take $m = 1$, then we get following inequalities for exponentially convex function:

$$\begin{aligned} 2e^{\eta\left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2}\right)} \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega}, \zeta_2}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) (\zeta_1; p) & \leq 2 \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega}, \zeta_2}^{\rho, r, q, c} e^{\eta \circ \kappa} e^{\gamma \circ \kappa} \right) (\zeta_1; p) \\ & \leq \left(e^{\eta(\kappa(\zeta_2))} + e^{\eta(\kappa(\zeta_1))} \right) \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega}, \zeta_2}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) (\zeta_1; p), \end{aligned} \tag{27}$$

where $\bar{\omega} = \frac{\omega}{(\kappa(\zeta_2) - \kappa(\zeta_1))^\theta}$.

- Remark 4.**
1. If we take $\kappa(u) = u$ in (25), then Theorem 7 is obtained.
 2. If we take $\kappa(u) = u$ and $m = 1$ in (25), then [15, Corollary 2.8] is obtained.
 3. If we take $\kappa(u) = u$ in (27), then [15, Corollary 2.8] is obtained.

In the following we give another generalized version of the Fejér-Hadamard inequality.

Theorem 11. Let $\eta, \kappa : [\zeta_1, m\zeta_2] \subset \mathbb{R} \rightarrow \mathbb{R}$, $0 < \zeta_1 < m\zeta_2$ be the real-valued functions. If η be a integrable, exponentially m -convex and $\eta(\kappa(v)) = \eta(\kappa(\zeta_1) + m\kappa(\zeta_2) - m\kappa(v))$ and κ be a differentiable and strictly increasing. Also, let $\gamma : [\zeta_1, m\zeta_2] \rightarrow \mathbb{R}$ be a function which is non-negative and integrable. Then for integral operators (9) and (10), the following inequalities hold:

$$\begin{aligned} & 2e^{\eta\left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2}\right)} \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} (2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m}\right)\right)}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p \right) \\ & \leq (1 + m) \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} (2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m}\right)\right)}^{\rho, r, q, c} e^{\eta \circ \kappa} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p \right) \\ & \leq \frac{m}{(m\kappa(\zeta_2) - \kappa(\zeta_1))} \left[\left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} (2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m}\right)\right)}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p \right) \right. \\ & \quad \left. + \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta\left(\frac{\kappa(\zeta_1)}{m^2}\right)} \right) (m\kappa(\zeta_2) - \kappa(\zeta_1)) \left({}_{\kappa} Y_{\theta, \theta, l, \bar{\omega} (2m)^\theta, \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m}\right)\right)}^{\rho, r, q, c} e^{\gamma \circ \kappa} \right) \left(\kappa^{-1}\left(\frac{\kappa(\zeta_1)}{m}\right); p \right) \right], \end{aligned} \tag{28}$$

where $\bar{\omega} = \frac{\omega}{(m\kappa(\zeta_2) - \kappa(\zeta_1))^\theta}$.

Proof. Multiplying both sides of (20) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma\left(\frac{\tau}{2}\kappa(\zeta_2) + \frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)}$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & 2e^{\eta\left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2}\right)} \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma\left(\frac{\tau}{2}\kappa(\zeta_2) + \frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)} d\tau \\ & \leq \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta\left(\frac{\tau}{2}\kappa(\zeta_1) + m\frac{(2-\tau)}{2}\kappa(\zeta_2)\right)} e^{\gamma\left(\frac{\tau}{2}\kappa(\zeta_2) + \frac{(2-\tau)}{2}\frac{\kappa(\zeta_1)}{m}\right)} d\tau \end{aligned}$$

$$+ m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} e^{\gamma \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} d\tau. \tag{29}$$

Putting $\kappa(v) = \frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m}$ in (29), we get

$$\begin{aligned} & 2e^{\eta \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2} \right)} \int_{\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right)}^{\kappa^{-1} \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m} \right)} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c} \left((2m)^\theta \bar{\omega}(\kappa(v) - \frac{\kappa(\zeta_1)}{m})^\theta; p \right) e^{\gamma(\kappa(v))} d(\kappa(v)) \\ & \leq \int_{\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right)}^{\kappa^{-1} \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m} \right)} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c} \left((2m)^\theta \bar{\omega}(\kappa(v) - \frac{\kappa(\zeta_1)}{m})^\theta; p \right) \\ & \quad \times e^{\eta(\kappa(\zeta_1) + m\kappa(\zeta_2) - m\kappa(v))} e^{\gamma(\kappa(v))} d(\kappa(v)) + m \int_{\kappa^{-1} \left(\frac{\kappa(\zeta_1)}{m} \right)}^{\kappa^{-1} \left(\frac{\kappa(\zeta_1) + m\kappa(\zeta_2)}{2m} \right)} \left(\kappa(v) - \frac{\kappa(\zeta_1)}{m} \right)^{\theta-1} \\ & \quad \times E_{\theta, \theta, l}^{\rho, r, q, c} \left((2m)^\theta \bar{\omega}(\kappa(v) - \frac{\kappa(\zeta_1)}{m})^\theta; p \right) e^{\eta(\kappa(v))} e^{\gamma(\kappa(v))} d(\kappa(v)). \end{aligned}$$

By using (10) and given condition $\eta(\kappa(v)) = \eta(\kappa(\zeta_1) + m\kappa(\zeta_2) - m\kappa(v))$, the first inequality of (28) is obtained.

Now multiplying both sides of (21) with $\tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)}$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta \left(\frac{\tau}{2} \kappa(\zeta_1) + m \frac{(2-\tau)}{2} \kappa(\zeta_2) \right)} e^{\gamma \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} d\tau \\ & + m \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\eta \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} e^{\gamma \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} d\tau \\ & \leq \left(e^{\eta(\kappa(\zeta_1))} - m^2 e^{\eta \left(\frac{\kappa(\zeta_1)}{m^2} \right)} \right) \int_0^1 \tau^\theta E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} d\tau \\ & + m \left(e^{\eta(\kappa(\zeta_2))} + m e^{\eta \left(\frac{\kappa(\zeta_1)}{m^2} \right)} \right) \int_0^1 \tau^{\theta-1} E_{\theta, \theta, l}^{\rho, r, q, c}(\omega \tau^\theta; p) e^{\gamma \left(\frac{\tau}{2} \kappa(\zeta_2) + \frac{(2-\tau)}{2} \frac{\kappa(\zeta_1)}{m} \right)} d\tau. \end{aligned}$$

From above the second inequality of (28) is achieved. \square

Corollary 4. Under the assumptions of Theorem 11 if we take $m = 1$, then we get following inequalities for exponentially convex function:

$$\begin{aligned} & 2e^{\eta \left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2} \right)} \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega} 2^\theta, \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2} \right) \right)}^{\rho, r, q, c} - e^{\gamma \circ \kappa} \right) (\zeta_1; p) \leq 2 \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega} 2^\theta, \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2} \right) \right)}^{\rho, r, q, c} - e^{\eta \circ \kappa} e^{\gamma \circ \kappa} \right) (\zeta_1; p) \\ & \leq \left(e^{\eta(\kappa(\zeta_2))} + e^{\eta(\kappa(\zeta_1))} \right) \left({}_\kappa Y_{\theta, \theta, l, \bar{\omega} 2^\theta, \left(\kappa^{-1} \left(\frac{\kappa(\zeta_1) + \kappa(\zeta_2)}{2} \right) \right)}^{\rho, r, q, c} - e^{\gamma \circ \kappa} \right) (\zeta_1; p), \end{aligned} \tag{30}$$

where $\bar{\omega} = \frac{\omega}{(\kappa(\zeta_2) - \kappa(\zeta_1))^\theta}$.

4. Concluding remarks

Here we have proved two generalized fractional versions of the Hadamard inequality as well as two generalized fractional versions of the Fejér-Hadamard inequality. For proving these fractional inequalities we have utilized exponentially m -convex functions and generalized integral operators containing Mittag-Leffler functions in their kernels. The presented results hold for exponentially convexity and well known fractional integral operators given in Remark 1. Reader can obtain desired fractional Hadamard and fractional Fejér-Hadamard inequalities from this paper.

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