Results on the Joint Essential Maximal Numerical Ranges

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Abstract

In the present paper, we show the equivalent definitions of the joint essential maximal numerical range. In the current paper, we show that the properties of the classical numerical range such as compactness also hold for the joint essential maximal numerical range. Further, we show that the joint essential maximal numerical range is contained in the joint maximal numerical range.

Keywords: Numerical range; essential maximal numerical range; maximal numerical range.

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1 Introduction and Preliminaries

We denote by $B(X)$ the algebra of (bounded) linear operators acting on complex Hilbert space X with inner product $\langle , \rangle$ and norm $\|.\|$. Stampfli in [1] introduced and studied the concept of maximal numerical range of a bounded operator T on $B(X)$ and used it to derive an identity for the norm of derivation. The maximal numerical range of an operator T is denoted by $\mathrm{Max}W(T)$ and defined as $\mathrm{Max}W(T) = \{r \in \mathbb{C} : \langle Tx_n, x_n \rangle \to r, \text{ where } x_n \in X; \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}$.

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Recall here that a derivation on a Hilbert space X is a linear transformation $\delta : X \to X$ that satisfies $\delta(xy) = x\delta(y) + \delta(x)y \ \forall x, y \in X$. Recall also that derivation δ is said to be an inner derivation if for a fixed x we have $\delta : y \to xy - yx$. For an operator $T \in B(X)$, the inner derivation is denoted and defined as $\delta_T(Y) = TY - YT$ where $Y \in B(X)$. Stampfli [1] determined the norm of an inner derivation and showed that $\|\delta_T\| = 2 \inf\{\|T - \lambda\| : \lambda \in \mathbb{C}\}$. Several other properties of the set $\mathrm{Max}W(T)$ are known. For instance, it is clear from the following theorem that the set $\mathrm{Max}W(T)$ is nonempty, closed and convex.

Theorem 1.1. $\mathrm{Max}W(T)$ *is nonempty, closed and convex subset of the closure of numerical range.*

The proof of the theorem can be found in Stampfli [1].

The concept of maximal numerical range was later generalised by Ghan in [2] to the joint maximal numerical range, $\mathrm{Max}W_m(T)$, of an $m-$tuple operator $T = (T_1, ..., T_m) \in B(X)$. The joint maximal numerical range of $T = (T_1, ..., T_m) \in B(X)$, denoted by $\mathrm{Max}W_m(T)$, is defined as,

$\mathrm{Max}W_m(T) = \{r \in \mathbb{C}^m : \langle T_k x_n, x_n \rangle \to r_k, \text{ where } x_n \in X; \|x_n\| = 1 \text{ and } \|T_k x_n\| \to \|T_k\|; 1 \le k \le m\}$.

In the case $k = 1$, it is the usual maximal numerical range of an operator T. From the properties of the joint maximal numerical range, it is known that $\mathrm{Max}W_m(T)$ does not have translation property by scalar, that is $\mathrm{Max}W_m(\beta T + \alpha I) \ne \beta \mathrm{Max}W_m(T) + \alpha \ \forall \ \beta, \alpha \in \mathbb{C}^m$.

In particular, it is known that $\mathrm{Max}W_m(T) \cap \mathrm{Max}W_m(T + \beta) = \emptyset$ for any $0 \ne \beta = (\beta_1, ..., \beta_m) \in \mathbb{C}^m$ (see [3]).

Several other properties of the set $\mathrm{Max}W_m(T)$ are also known as shown by the following results.

Theorem 1.2. *The following conditions are equivalent for an operator* $T = (T_1, ..., T_m) \in B(X)$.

(i) $0 \in MaxW_m(T)$

(ii) $\|T\|^2 + |r|^2 \le \|T\|^2 \ \forall \ r = (r_1, ..., r_m) \in \mathbb{C}^m$

(iii) $\|T\| \le \|T + r\| \ \forall \ r = (r_1, ..., r_m) \in \mathbb{C}^m$

The proof of the theorem can be found in Khan [2].

Lemma 1. *Suppose* $T = (T_1, ..., T_m) \in B(X), \|T_k\| = 1$ *and* $\|x_n\| = 1$.

If $\|T_k x_n\|^2 \ge (1 - \epsilon)$, *then* $\|(T_k^* T_k - I) x_n\|^2 \le 2\epsilon$.

Proof. Since $T_k^* T_k - I \ge 0$ it follows that,

$$
\begin{aligned}
\|(T_k^* T_k - I) x_n\|^2 &= \|T_k^* T_k x_n\|^2 - 2\|T_k x_n\|^2 + \|x_n\|^2 \\
&= (\|T_k^* T_k x_n\|^2 - \|T_k x_n\|^2) - (\|T_k x_n\|^2 - \|x_n\|^2) \\
&\le 2\epsilon.
\end{aligned}
$$

$\square$

It is known that $\mathrm{Max}W_m(T)$ is compact but is not always nonempty for

$T = (T_1, ..., T_m) \in B(X)$. It is also not always convex. In the following theorem, we use the classical Toeplitz-Hausdorff Theorem to show one case in which $\mathrm{Max}W_m(T)$ is convex.

Theorem 1.3. *Let* $T = (T_1, ..., T_m) \in B(X)$ *be an* $m-$tuple *of operators. The set* $MaxW_m(T)$ *is convex.*

Proof. Let $\lambda = (\lambda_1, ..., \lambda_m)$, $\mu = (\mu_1, ..., \mu_m) \in \mathrm{Max}W_m(T)$.

Since $\lambda = (\lambda_1, ..., \lambda_m)$, $\mu = (\mu_1, ..., \mu_m) \in \mathrm{Max}W_m(T)$, it implies that there exist $x_n, y_n \in X$ such that $\|x_n\| = 1, \|T_k x_n\| \to \|T_k\|, \langle T_k x_n, x_n \rangle \to \lambda;\ 1 \le k \le m$
and $\|y_n\| = 1, \|T_k y_n\| \to \|T_k\|, \langle T_k y_n, y_n \rangle \to \mu;\ 1 \le k \le m$.

Let M_n be a subspace spanned by x_n and y_n and $\mathcal{P}_n$ be a projection of X onto M_n. Suppose $T_n = \mathcal{P}_n T_k \mathcal{P}_n$, then $\langle T_k x_n, x_n \rangle = \langle T_k y_n, y_n \rangle$ are in the numerical range of $\mathcal{P}_n T_k \mathcal{P}_n$. By Toeplitz-Hausdorff Theorem, $W(\mathcal{P}_n T_k \mathcal{P}_n)$ is convex and so for each n we can choose α_n, β_n with $\nu_n = \alpha_n x_n + \beta_n y_n = 1$ (where ν_n is a sequence in X). If η is a point on the line segment joining $\lambda = (\lambda_1, ..., \lambda_m)$ and $\mu = (\mu_1, ..., \mu_m)$ then $\langle T_k \nu_n, \nu_n \rangle \to \eta$ and $\|\nu_n\| = 1$. Note that $|\langle x_n, y_n \rangle| \le \theta < 1$ for n sufficiently large. This implies that the angle between x_n and y_n is bounded away from 0. Therefore, there exists a constant M such that $|\alpha_n| \le M$ and $|\beta_n| \le M$ for n sufficiently large, where $\|\nu\| = \|\alpha_n x_n + \beta_n y_n\| = 1$. By Lemma 1, $\| T_k \nu_n \| = \langle T_k^* T_k \nu_n, \nu_n \rangle = \langle T_k^* T_k (\alpha_n x_n + \beta_n y_n), (\alpha_n x_n + \beta_n y_n) \rangle = \|\nu_n\|^2 - 2M\epsilon$ where $\epsilon_n \to 0$. That is, $\| (T_k^* T_k - I) x_n \| \to 0$ and $\| (T_k^* T_k - I) y_n \| \to 0$ as $n \to \infty$. Thus $\|T_k \nu_n\| \to 1$ as $n \to \infty$ implying that $\|T \nu_n\| \to \|T\|$ as $n \to \infty$. $\qquad\square$

We also recall that $\mathrm{Max}W_m(T)$ corresponds to the joint numerical range produced by maximal vectors (vectors x such that $\|x_n\| = 1$ and $\|Tx_n\| = \|T\| \cdot \|x_n\| = \|T\|$) when X is finite dimensional. See [2] for this and more.

Recall here that the joint numerical range of an m-tuple operator $T = (T_1, ..., T_m) \in B(X)$ is denoted and defined as,

$$W_m(T) = \left\{ \left(\langle T_1 x, x \rangle, ..., \langle T_m x, x \rangle \right) : x \in X, \langle x, x \rangle = 1 \right\}.$$

This study of joint numerical range of an m-tuple operator $T = (T_1, ..., T_m) \in B(X)$ was generalised to the study of the joint numerical range of the Aluthge transform $\widetilde{T}$ of an m-tuple operator $T = (T_1, ..., T_m)$ in [4]. This notion was also generalised to the study of the joint essential numerical range of Aluthge transform $\widetilde{T}$ of an m-tuple operator $T = (T_1, ..., T_m)$ in [5]. Here, the Aluthge transform $\widetilde{T}$ of the operator T is defined as the operator $T = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ where $T = U|T|$ is any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$.

2 Joint Essential Maximal Numerical Range

Fong [6] introduced the essential maximal numerical range to study the norm of a derivation on Calkin algebra in 1997. The essential maximal numerical range, $\mathrm{Max}W_e(T)$, is defined as

$$\mathrm{Max}W_e(T) = \{r \in \mathbb{C} : \langle Tx_n, x_n \rangle \to r,\ x_n \to 0 \text{ weakly and } \|Tx_n\| \to \|T\|_e\}.$$

Here, $\|T\|_e$ denotes the essential norm of T defined by $\|T\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(X)\}$ where $\mathcal{K}(X)$ is the ideal of all compact operators in $B(X)$.

It is clear that $\|\delta_t\| = 2\|t\|$ if and only if $0 \in \mathrm{Max}W_e(T)$ where t is the image of T in the Calkin algebra. See [6].

Lemma 2. $\mathrm{Max}W_e(T)$ *is nonempty, closed and convex subset of the essential numerical range.*

See Fong [6] for the proof.

In the following theorem, $\mathcal{R}e\lambda$ stands for the real part of $\lambda \in \mathbb{C}$ while $Im\lambda$ stands for the imaginary part of $\lambda \in \mathbb{C}$. Here, $\mathcal{R}e(T) = \frac{1}{2}(T + T^*)$ while $Im(T) = \frac{1}{2i}(T - T^*)$ where, for an operator $T \in B(X)$, T^* denotes its adjoint. We remind the reader that the adjoint of an operator $T \in B(X)$ is a linear operator $T^* \in B(X)$ defined by the relation $\langle Tx, y \rangle = \langle x, T^*y \rangle \ \forall y, x \in X$.

Theorem 2.1. *Suppose $T \in B(X)$. Then $\mathrm{Max}W_e(T) \cap \mathrm{Max}W_e(T - \alpha) = \emptyset$ for $0 \neq \alpha \in \mathbb{C}$.*

Proof. Let $\mu \in \mathrm{Max}W_e(T) \cap \mathrm{Max}W_e(T - \alpha)$. Then $\mu \in \mathrm{Max}W_e(T)$ and $\mu \in \mathrm{Max}W_e(T - \alpha)$. By definition, when $\mu \in \mathrm{Max}W_e(T)$, then there exist a sequence $\{x_n\} \in X$ of unit vectors converging weakly to $0 \in X$ such that $\langle Tx_n, x_n \rangle \to r$ and $\|Tx_n\| \to \|T\|_e$ for $r \in \mathbb{C}$.

Also, for $\alpha \in \mathbb{C}$, when $\mu \in \mathrm{Max}W_e(T - \alpha)$ then there exists a sequence $\{x_n\} \in X$ of unit vectors converging weakly to $0 \in X$ such that $\langle (T - \alpha)x_n, x_n \rangle \to r$ and $\|(T - \alpha)x_n\| \to \|T - \alpha\|_e$ for $r \in \mathbb{C}$. Now, Since $\mu \in \mathrm{Max}W_e(T)$, then, for $r \in \mathbb{C}$ we have

$$\|(T + r)x_n\|^2 = \|Tx_n\|^2 + 2\mathcal{R}e\, \bar{r}\langle Tx_n, x_n \rangle + |r|^2.$$

From this, we have

$$\|T\|^2 + 2\mathcal{R}e\, \bar{r}\mu + |r|^2 \leq \|T + r\|^2. \tag{1}$$

Also, since $\mu \in \mathrm{Max}W_e(T - \alpha)$, then for $r \in \mathbb{C}$ we have

$$\| [(T - \alpha) + r] x_n\|^2 = \|(T - \alpha)x_n\|^2 + 2\mathcal{R}e\, \bar{r}\langle (T - \alpha)x_n, x_n \rangle + |r|^2$$

which implies that

$$\|T - \alpha\|^2 + 2\mathcal{R}e\, \bar{r}\mu + |r|^2 \leq \|T - \alpha + r\|^2. \tag{2}$$

Letting $r = -\alpha$ and $r = \alpha$ in (1) and (2) respectively, we obtain

$$\|T\|^2 - 2\mathcal{R}e\, \bar{\alpha}\mu + |\alpha|^2 \leq \|T - \alpha\|^2$$

and

$$\|T - \alpha\|^2 + 2\mathcal{R}e\, \bar{\alpha}\mu + |\alpha|^2 \leq \|T\|^2.$$

Combining these two results yield $|\alpha|^2 \leq 0$ which is impossible. Thus the assumption is wrong. $\square$

The notion of essential maximal numerical range was generalised to the study of the joint essential maximal numerical range by Khan [3] who proved certain results analogous to the single operator case. The joint essential maximal numerical range, denoted by $\mathrm{Max}W_{e_m}(T)$, is defined as

$$\mathrm{Max}W_{e_m}(T) = \{r \in \mathbb{C}^m : \langle T_k x_n, x_n \rangle \to r_k, x_n \to 0 \text{ weakly and } \|T_k x_n\| \to \|T_k\|_e; 1 \leq k \leq m\}.$$

Here, $\|T_k\|_e$ denotes the essential norm of T_k defined by $\|T_k\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(X)\}$. In [3], it was shown that $\mathrm{Max}W_{e_m}(T) \cap \mathrm{Max}W_{e_m}(T + \beta) = \emptyset$ for $0 \neq \beta = (\beta_1, ..., \beta_m) \in \mathbb{C}^m$. In the case $k = 1$, the joint essential maximal numerical range becomes the usual essential maximal numerical range, $\mathrm{Max}W_e(T)$.

The joint essential maximal numerical range was also studied by Cyprian, Masibayi and Okelo [7] who together showed that it is convex among other interesting results. In [8], Cyprian generalised the notion of the joint essential maximal numerical range to the study on the joint maximal numerical range of aluthge transform $\widetilde{T}$ of an m-tuple operator $T = (T_1, ..., T_m)$.

This paper is a continuation of the study of the notion of the joint essential maximal numerical range which is helpful in the development of the research on numerical ranges. We examine some properties of the set $\mathrm{Max}W_{e_m}(T)$. The following theorem proves some equivalent definitions of the joint essential maximal numerical range.

Theorem 2.2. *Suppose X is an infinite dimensional complex Hilbert space and $T = (T_1, ..., T_m) \in B(X)$. Let $r = (r_1, ..., r_m) \in \mathbb{C}^m$. The following properties are equivalent:*

    1. $r \in \mathrm{Max}W_{e_m}(T)$

    2. There exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$$\langle\, T_k x_n, x_n \rangle \to r_k \ \text{and} \ \|T_k x_n\| \ \to \ \|T_k\|_e \ ; 1 \le \ k \le \ m.$$

3. *There exists a sequence $\{x_n\}_{n=1}^{\infty} \in X$ of vectors converging weakly to $0 \in X$ such that*

$$\langle\, T_k x_n, x_n \rangle \to r_k \ \text{and} \ \|T_k x_n\| \ \to \ \|T_k\|_e \ ; 1 \le \ k \le \ m.$$

4. *There exists an infinite - dimensional projection P such that*

$$P(T_k - r_k I)P \in \mathcal{K}(X) \ \text{and} \ \|T_k P\|_e \ = \ \|T_k\|_e \ ; k = 1, ..., m.$$

Proof. Note that $1 \Leftrightarrow 2$ and $1 \Leftrightarrow 4$ was proved by Khan [3].

To prove that $2 \Rightarrow 4$, let $\{x_n\}_{n=1}^{\infty} \in X$ be an orthonormal sequence such that

$$\langle\, T_k x_n, x_n \rangle \to r_k \ \text{and} \ \|T_k x_n\| \ \to \ \|T_k\|_e \ ; 1 \le \ k \le \ m.$$

By passing to a subsequence we can assume that

$$\sum_{n=1}^{\infty} |\langle (T_k - r_k)x_n, x_n \rangle|^2 < \infty \tag{3}$$

Let $n_1 = 1$. Then $\sum_{n=1}^{\infty} |\langle (T_k - r_k)x_{n_1}, x_n \rangle|^2 \le \|(T_k - r_k)x_{n_1}\|^2$

and $\sum_{n=1}^{\infty} |\langle (T_k - r_k)x_n, x_{n_1} \rangle|^2 \le \|(T_k - r_k)^* x_{n_1}\|^2$. Thus, by Bessels inequality, there is an integer

$n_2 > n_1$ such that $\sum_{n=n_2}^{\infty} |\langle (T_k - r_k)x_{n_1}, x_n \rangle|^2 < 2^{-1}$ and

$\sum_{n=n_2}^{\infty} |\langle (T_k - r_k)x_n, x_{n_1} \rangle|^2 < 2^{-1}$. If this procedure is repeated, a strictly increasing sequence $\{n_t\}_{t=1}^{\infty}$ of positive integers is obtained such that we have

$$\sum_{n=n_{t+1}}^{\infty} |\langle (T_k - r_k)x_{n_t}, x_n \rangle|^2 < 2^{-t}$$

and

$$\sum_{n=n_{t+1}}^{\infty} |\langle (T_k - r_k)x_n, x_{n_t} \rangle|^2 < 2^{-t} \tag{4}$$

Both (3) and (4) imply that

$$\sum_{t,\, l\,=1}^{\infty} |\langle (T_k - r_k)x_t, x_{n_l} \rangle|^2 < \infty \tag{5}$$

If P is an orthogonal projection onto the subspace $\mathcal{M}$ spanned by $x_{n_1}, x_{n_2}, ...,$ then
$\sum_{t,\, l\,=1}^{\infty} |\langle (PT_k P - r_k P)x_{n_t}, x_{n_l} \rangle|^2 = \sum_{t,\, l\,=1}^{\infty} |\langle (T_k - r_k)x_{n_t}, x_{n_l} \rangle|^2 < \infty$ by (5), hence $PT_k P$ is a Hilbert - Schmidt operator and therefore $PT_k P - r_k P \in K(X)$.

We now show that (3) implies (2). Let $\{x_n\}_{n=1}^{\infty} \in X$ be a sequence of vectors converging weakly to $0 \in X$ such that $\langle\, T_k x_n, x_n \rangle \to r_k$ and $\|T_k x_n\| \ \to \ \|T_k\|_e \ ; 1 \le \ k \le \ m$.

Construct an orthonormal sequence $\{y_n\}_{n=1}^{\infty}$ such that $\|T_k y_n\| \to \|T_k\|_e - \frac{1}{n}$ and $|(T_k y_n, y_n)| < \frac{1}{n}$ as follows. Suppose that the set $\{x_1, ..., x_n\}$ has been constructed. Let $\mathcal{M}$ be the subspace spanned by $x_1, ..., x_n$ and P be the projection onto $\mathcal{M}$. Then we have $\|P x_n\| \to 0$ as $n \to \infty$.

Let $z_n = \left\| (I-P)x_n \right\|^{-1} \left((I-P)x_n \right)$.

We have $T_k z_n = \left\| (I-P)x_n \right\|^{-1} \left(T_k(I-P)x_n \right)$. This gives

$$\langle T_k z_n, z_n \rangle = \left\langle \left\| (I-P)x_n \right\|^{-1} \left(T_k(I-P)x_n \right), \left\| (I-P)x_n \right\|^{-1} \left(T_k(I-P)x_n \right) \right\rangle$$
$$= \left\| (I-P)x_n \right\|^{-2} \left\{ \langle T_k x_n, x_n \rangle - \langle T_k x_n, P x_n \rangle - \langle T_k P x_n, x_n \rangle + \langle T_k P x_n, P x_n \rangle \right\}$$
$$\to r_k.$$

We choose n large enough such that $|\langle T_k z_n, z_n \rangle - r_k| < (n+1)^{-1}$. If we let $z_n = x_{n+1}$ we get $|\langle T_k x_{n+1}, x_{n+1} \rangle - r_k| < (n+1)^{-1}$

To show that (3) implies (1), suppose that for a point $r_k \in \mathbb{C}^m$ there is a sequence $\{x_n\} \in X$ such that $\langle T_k x_n, x_n \rangle \to r_k$. Since every sequence $\{x_n\} \to 0$ weakly, and $\|x_n\| = 1$, we have $r_k \to \mathrm{Max}W_{e_m}(T)$. $\qquad \square$

We state the following theorem without proof since its proof runs like that of Theorem 2.1

Theorem 2.3. *Let $T = (T_1, ..., T_m) \in B(X)$. Then $\mathrm{Max}W_{e_m}(T) \cap \mathrm{Max}W_{e_m}(T-\alpha) = \emptyset$ for $0 \neq \alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$.*

Theorem 2.4. *Let $T = (T_1, ..., T_m) \in B(X)$. If $0 \in \mathrm{Max}W_{e_m}(T-\alpha)$, then $\|T-\alpha\|^2 + |\alpha|^2 \le \|T\|^2$ for all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$.*

Proof. Suppose $0 \in \mathrm{Max}W_{e_m}(T-\alpha)$, then there exists a sequence $x_n \in X$ such that $\langle (T_k - \alpha_k)x_n, x_n \rangle \to 0$, where $x_n \in X$; $\|x_n\| = 1$ and $\|(T_k - \alpha_k)x_n\| \to \|(T_k - \alpha_k)\|$; $1 \le k \le m$. But $\|T_k - \alpha_k\|^2 + |\alpha_k|^2 = \lim_{n \to \infty} \|T_k x_n\|^2 \le \|T_k\|^2$; $1 \le k \le m$. Taking finite summation on both sides we obtain $\sum_{k=1}^{m} \|T_k - \alpha_k\|^2 + \sum_{k=1}^{m} |\alpha_k|^2 \le \sum_{k=1}^{m} \|T_k\|^2$. Hence, $\|T-\alpha\|^2 + |\alpha|^2 \le \|T\|^2$ for all $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$. $\qquad \square$

Recall that a subset C of a linear space M is convex if $\forall x, y \in C$ the segment joining x and y is contained in C, that is, $tx + (1-t)y \in C$ $\forall t \in [0,1]$. A set S is starshaped if $\exists\ y \in S$ such that $\forall x \in S$ the segment joining x and y is contained in S, that is $\lambda x + (1-\lambda)y \in S$ $\forall \lambda \in [0,1]$.

A point $y \in S$ is a star center of S if there is a point $x \in S$ such that the segment joining x and y is contained in S.

Starshapedness is related to convexity in that a convex set is starshaped with all its points being star centers.

Theorem 2.5. *Suppose $T = (T_1, ..., T_m) \in B(X)$. Then $\mathrm{Max}W_{e_m}(T)$ is nonempty, compact and each element $r \in \mathrm{Max}W_{e_m}(T)$ is a star center of $\mathrm{Max}W_m(T)$. Moreover, $\mathrm{Max}W_{e_m}(T)$ is convex.*

Proof. First, we prove that $\mathrm{Max}W_{e_m}(T)$ is nonempty. To do this, from Theorem 2.2, there exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \in X$ such that

$\langle T_k x_n, x_n \rangle \to r_k$ and $\|T_k x_n\| \to \|T_k\|_e$; $1 \le k \le m$.

Thus the sequence $\{\langle T_k x_n, x_n \rangle\}_{n=1}^{\infty}$ is bounded. Choose a subsequence and assume that $\langle T_k x_n, x_n \rangle$ converges. Then $\mathrm{Max}W_{e_m}(T)$ is nonempty.

The compactness of $\mathrm{Max}W_{e_m}(T)$ can be seen right from its properties. That is, $\mathrm{Max}W_{e_m}(T) = \mathrm{Max}W_{e_m}(T + K) \subseteq \mathrm{Max}W_m(T + K) : K \in \mathcal{K}(X)$. Since $\mathrm{Max}W_m(T + K)$ compact, the joint essential numerical range is also compact.

To prove that each element $r \in \mathrm{Max}W_{e_m}(T)$ is a star center of $\mathrm{Max}W_m(T)$, it should be shown that $(1-\lambda)p + \lambda r \in \mathrm{Max}W_m(T) : \lambda \in [0, 1]$ where $r \in \mathrm{Max}W_{e_m}(T)$ and $p \in \mathrm{Max}W_m(T)$. Assume without loss of generality that $\|T\| = 1$. Suppose $s \in \mathrm{Max}W_m(T)$ so that $s = \lambda r + (1 - \lambda)p$. Let $\{x_n\}$ and $\{e_n\}$ be orthonormal sequences in X such that $r = \langle Tx_n, x_n \rangle, p = \langle Te_n, e_n \rangle$ and $\|x_n\| = \|e_n\| = 1$. Then,

$$
\begin{aligned}
s &= \lambda \langle Tx_n, x_n \rangle + (1 - \lambda)\langle Te_n, e_n \rangle \\
&= \left\langle T\sqrt{\lambda}\, x_n, \sqrt{\lambda}\, x_n \right\rangle + \left\langle T\sqrt{1-\lambda}\, e_n, \sqrt{1-\lambda}\, e_n \right\rangle \\
&= \left\langle (T\sqrt{\lambda}\, x_n + T\sqrt{1-\lambda}\, e_n), (\sqrt{\lambda}\, x_n + \sqrt{1-\lambda}\, e_n) \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left\| \sqrt{\lambda}\, x_n + \sqrt{1-\lambda}\, e_n \right\|^2 &= \left(\left\| \sqrt{\lambda}\, x_n \right\|^2 + \left\| \sqrt{1-\lambda}\, e_n \right\|^2 \right) \\
&= \lambda \|x_n\|^2 + (1-\lambda)\|e_n\|^2 \\
&= \lambda + (1 - \lambda) = 1
\end{aligned}
$$

Thus, $(1 - \lambda)r + \lambda p \in \mathrm{Max}W_m(T)$.

Convexity of $\mathrm{Max}W_{e_m}(T)$ is proved by showing that for $r, p \in \mathrm{Max}W_{e_m}(T)$ and $\lambda \in [0, 1]$ we have $\lambda r + (1 - \lambda)p \in \mathrm{Max}W_{e_m}(T)$. Now, $r \in \mathrm{Max}W_{e_m}(T) = \mathrm{Max}W_{e_m}(T + K)$ for every $K \in \mathcal{K}(X)$ and $p \in \mathrm{Max}W_{e_m}(T) = \mathrm{Max}W_{e_m}(T + K) \subseteq \mathrm{Max}W_m(T + K)$.
By Theorem 2.5, $\lambda r + (1 - \lambda)p \in \mathrm{Max}W_m(T + K)$.
Thus, $\lambda r + (1 - \lambda)p \in \cap\{\mathrm{Max}W_m(T + K) : K \in \mathcal{K}(X)\} = \mathrm{Max}W_{e_m}(T)$. Hence $\mathrm{Max}W_{e_m}(T)$ is convex. $\qquad\square$

The following theorem shows the relation between the sets $\mathrm{Max}W_{e_m}(T)$ and $\mathrm{Max}W_m(T)$. Here, we let $T = U|T|$ be any polar decomposition of T with U a partial isometry and $|T| = (T^*T)^{\frac{1}{2}}$, where, for an operator $T \in B(X)$, T^* denotes its adjoint.

Theorem 2.6. *Let $T \in B(X)$ and $|T| = (T^*T)^{\frac{1}{2}}$. Then $\mathrm{Max}W_{e_m}(T) \subseteq \mathrm{Max}W_m(T)$.*

Proof. Assume, without loss of generality that $\|T\| = 1$ and let $r = (r_1, r_2, ..., r_m) \in \mathrm{Max}W_{e_m}(T)$. Then, there exists a sequence $\{x_n\} \in X$ of unit vectors converging weakly to $0 \in X$ such that $\langle Tx_n, x_n \rangle \to r$ and $\|Tx_n\| \to \|T\|_e$. Then, $\||T|^{1/2}x_n\| = \||T|^{1/2}\| = 1$ as $n \to \infty$ and $\|(1 - |T|)x_n\| = 0$ as $n \to \infty$. Also, $\|(1 - |T|^3)x_n\| = 0$ as $n \to \infty$.
Thus, $\lim_{n\to\infty} \||T||T|^{1/2}x_n\| = \lim_{n\to\infty} \langle |T|^3 x_n, x_n \rangle = 1 = \|T\|$. And,

$$
\begin{aligned}
\lim_{n\to\infty} \left| \langle Tx_n, x_n \rangle - \langle T|T|^{\frac{1}{2}}x_n, |T|^{\frac{1}{2}}x_n \rangle \right| &= \lim_{n\to\infty} \left| \langle U|T|^{\frac{1}{2}}x_n, |T|^{\frac{1}{2}}x_n \rangle - \langle T|T|^{\frac{1}{2}}x_n, |T|^{\frac{1}{2}}x_n \rangle \right| \\
&= \lim_{n\to\infty} \left| \left\langle (U|T|^{1/2} - U|T||T|^{1/2})x_n, |T|^{1/2}x_n \right\rangle \right| \\
&= \lim_{n\to\infty} \left| \langle (U|T|^{1/2})(1 - |T|)x_n, |T|^{1/2}x_n \rangle \right| \\
&\leq \lim_{n\to\infty} \|U|T|^{1/2}\| \|(1 - |T|)x_n\| \||T|^{1/2}x_n\| \\
&= 0.
\end{aligned}
$$

If we let $z_n = (|T|^{1/2}x_n)/(\||T|^{1/2}x_n\|)$ then $\{z_n\} \in X$ is a sequence of unit vectors converging weakly to $0 \in X$ such that $\langle\, Tz_n, z_n \rangle \to r$ and $\|Tz_n\| \to \|T\|_e$. Thus $r \in \mathrm{Max}W_m(T)$. Hence $\mathrm{Max}W_{e_m}(T) \subseteq \mathrm{Max}W_m(T)$. $\qquad\square$

3   Conclusions

Section 2, studied the properties of the joint essential maximal numerical range. For instance, we proved the equivalent definitions of the joint essential maximal numerical range and proved that the set $\mathrm{Max}W_{e_m}(T)$ is nonempty, compact and convex. It was also shown that the set $\mathrm{Max}W_{e_m}(T)$ is a contained in the set $\mathrm{Max}W_m(T)$.

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Competing Interests

Author has declared that no competing interests exist.

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