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A Rigorous Homogenization for a Two-Scale Convergence Approach to Piping Flow Erosion with Deposition in a Spatially Heterogeneous Soil

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Authors' contributions

This work was carried out in collaboration among all authors. AS(PhD) designed the study, performed the numerical analysis, wrote the protocol and wrote the first draft of the manuscript. PAY(PhD) and IKD(Prof.) managed the technicalities in the analyses of the study and further guided the research. All authors read and approved the final manuscript.

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Abstract

We present a rigorous homogenization approach to modelling piping flow erosion in a spatially heterogeneous soil. The aim is to provide a justification to a formal homogenization approach to piping flow erosion with deposition in a spatially heterogeneous soil. Under the assumption that the soil domain is perforated periodically with cylindrical repeating microstructure, we begin by proving that a solution to the proposed set of microscopic equations exist. Two-scale convergence is then used to study the asymptotic behaviour of solutions to the microscopic problem as the microscopic length scale approaches zero(0). We thus derive rigorously a homogenized model or macro problem as well as explicit formula for the effective coefficients. A strong observation from the numerical simulation was that, soil particle concentration in the water/soil mixture decreases but at a decreasing rate whereas soil particle deposition increases at regions with increasing amount of particle concentration in the flow causing a reduction in bare pore spaces across the soil domain.

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1 Introduction

Several studies on porous media over the decades have been conducted and a lot of papers have been published, addressing the basics of flow and transport through porous media. In most cases averaging and the notion of representative elementary volume is used [1]. A number of studies [2, 3] on conceptual models and the theory of mixtures have also been proposed.

Normally, the heterogeneous medium is described as a medium with local parameters that can be described by functions rapidly varying with respect to space variables and time. Homogenization is an approach that allows us to study the macro behaviour of a medium by its micro properties, it therefore seeks to replace a heterogeneous material by an equivalent homogeneous one.

One of the early studies was conducted by [4] and [5]. They respectively studied effective conductivity of a media with small concentrations of randomly and periodically arranged inclusions. Effective viscosity of suspensions in compressible viscous fluids was also investigated by [6]. A general approach based on asymptotic tools which can deal with a variety of different physical problems was also introduced later by [7].

The homogenization procedure is in two forms, namely the formal homogenization and rigorous homogenization. The former is based on construction of asymptotic expansions using multiple scales. At least two natural spatial length scales are introduced. One measuring variations within one period cell (the fast scale) and the other measuring variations within the domain of interest (the slow scale). The effectiveness of the use of multiple space scales to treat systematically boundary value problems with rapidly varying periodic structure was established by [8].

The latter is also based on energy estimates. Since the coefficients of the equations involved are rapidly oscillating and the derivatives are multiplied by the characteristic length scale ϵ which is the ratio of the microscopic length scale to the macroscopic length scale, obtaining estimates independent of ϵ is very difficult. To achieve this however, one must pass to the limit in weak sense by using integration by parts and suitable test functions [9, 10]. This process acts as a rigorous justification of results normally obtained using the formal homogenization [11, 12].

In this paper we model the piping flow erosion phenomena in a spatially heterogeneous soil and justify mathematically by rigorous use of periodic homogenization thus showing that the results obtained from formal homogenization can be verified through this rigorous homogenization process via two scale convergence.

In what follows we give a few mathematical theorems and lemma [12] useful in the quest to rigorously obtain a homogenized model for the piping flow erosion via two-scale convergence.

1.1 Important theories on homogenization

Lemma 1.1. *For a smooth Y-periodic function G(y),*

$$\int_Y \frac{\partial G(y)}{\partial y_i} dy = 0, \quad i = 1, 2, 3, ..., d$$

Given a differential operator $A_0 = -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)$ and an equation of the form

$$A_0 u = B, \;\; u\, is\, Y - periodic, \quad \text{Lemma 1.2 hold} \tag{1.1}$$

Lemma 1.2. *For the existence of a solution to* (1.1)*, a necessary condition is*

$$\int_Y B(y)dy = 0$$

Lemma 1.2 is a solvability condition for (1.1). This plays an important role in homogenization and implications of which is analyzed in Proposition 1.1

Proposition 1.1. *The homogeneous equation*

$$A_0 u = 0 \tag{1.2}$$

have solutions that are constants in y.

Consequently it can be deduced from Proposition 1.1 *that all solutions to* (1.1) *differ by a constant in y.* let $\Lambda_T = \Lambda \times (0, T)$ for $T \in (0, \infty)$ and $C^\infty_\#(Y)$ a space of infinitely differentiable functions in $\mathbb{R}^2$ which are $Y-$ *periodic.*

Definition 1.1. Denote by (u^ϵ) a sequence of functions in $L^2(\Lambda_T)$, we say that (u^ϵ) converges two scale to a unique function $u_0(t, x, y) \in L^2(\Lambda_T \times Y)$ if and only if for any $\Psi \in L^2(\Lambda_T, C^\infty_\#(Y))$ we have

$$\lim_{\epsilon \to 0} \int_0^T \int_\Lambda u^\epsilon(t, x)\Psi(t, x, \frac{x}{\epsilon})dxdt = \frac{1}{|Y|} \int_0^T \int_\Lambda \int_Y u_0(t, x, y)\Psi(t, x, y)dydxdt$$

Theorem 1.3. *For a bounded sequence $u^\epsilon \in L^2(\Lambda_T)$, there exist a subsequence and a function $u_0 \in L^2(\Lambda_T \times Y)$ such that u^ϵ two scale converges to u_0. However, u^ϵ converges weakly in $L^2(\Lambda_T)$ to the average of the two scale limit over the unit cell :*

$$u^\epsilon \to \int_Y u_0(.,.,y)dy, \ \textit{weakly in} \ L^2(\Lambda_T)$$

2 Rigorous Homogenization of the Microscopic Problem via Two-scale Convergence

2.1 Choice of the micro structure and the microscopic problem

A bounded heterogeneous soil structure Λ in $\mathcal{R}^2$ of coordinates $x = (x_1, x_2)$ with periodic positioning of pores through which a water/soil particles mixture of volume Λ_f flowing through a soil matrix Λ_s with a purely geometrical fluid/soil interface $\partial\Lambda_s$ of no thickness. The spatial variable x is a macroscale (global) variable. We take in the space $\mathcal{R}^2$ a unit cell Y with a microscale (local) variable $y = (y_1, y_2)$ and define a characteristic length scale $\epsilon = \dfrac{l}{L}$ with $y_i = \frac{x_i}{\epsilon}$ for $i \in \{1, 2\}$, l and L denotes the characteristic length of the unit cell Y and the soil domain Λ. The reference unit cell Y has two pairwise disjoint connected domains Y^s and Y^f with smooth fluid/soil boundary ∂Y^s. A repeating arrangement of copies ϵY occupying the entire region Λ as shown in Fig. 1 was created.

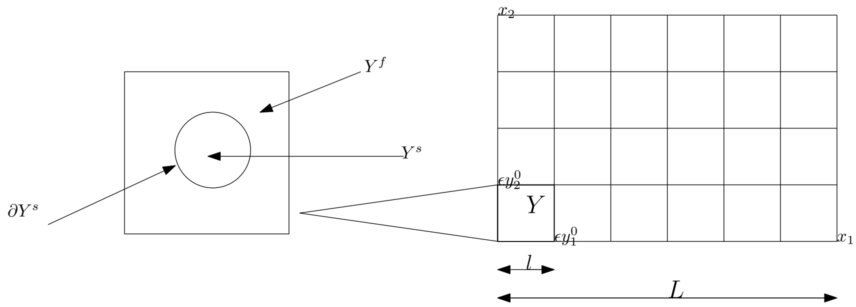

Fig. 1. (a) Left: Reference unit cell Y. (b) Right: Micro-scale geometry of the soil domain Λ^ϵ

2.2 Function spaces and unknowns of the microscopic problem P^ϵ

We define a parameterized domain Λ^ϵ and discuss briefly the function spaces and few technical assumptions on the unknown objects in the microscopic model.

2.2.1 Function spaces

For $\epsilon = \dfrac{l}{L} > 0$ as in Fig. 1, we define $F = (0, T)$ for $T \in (0, \infty)$. Also for any $(t, x) \in F \times \mathbb{R}^2$, let $D^\epsilon(t, \dfrac{x}{\epsilon}) = D(t, y)$, $\mu^\epsilon(t, \dfrac{x}{\epsilon}) = \mu(t, y)$, $E_u^\epsilon(t, \dfrac{x}{\epsilon}) = E_u(t, y)$. The notations used in this section are listed as follows:

$\langle af, bg\rangle_{L^2(\Lambda^\epsilon)} = ab \int_{\Lambda^\epsilon} f(x)g(x)dx$ - inner product in $L^2(\Lambda^\epsilon)$ for $a, b \in \mathbb{R}$,
$||g||_{L^2(\Lambda^\epsilon)} = \sqrt{\int_{\Lambda^\epsilon} |g(x)|^2 dx}$- $L^2(\Lambda^\epsilon)$ norm of g,
$W = \{u \in H^1_\#(Y) \mid \int_Y u dy = 0\}$
$C^\infty_\#(Y)$ - space of infinitely differentiable functions in $\mathbb{R}^2$ which are $Y - periodic$,
$L^2_\#(Y)$ - $L^2 - norm$ on Y-periodic functions,
$H^1_\#(Y)$ - H^1 on functions that are $Y - periodic$,
$L^\infty(\Lambda)$ - $\{g|g : \Lambda \to \mathbb{R}$, g measurable such that there exists a $k \in \mathbb{R}$ with $|g| \leq k$, almost everywhere on $\Lambda\}$,
$||v||^p_{L^p(\Lambda;L^2(Y))} = \int_\Lambda ||v||^p_{L^2(Y)} dx$, for $p \in [1, \infty)$.
$W^{d,p}$ a Sobolev Space endowed with $d - order$ derivative and an $L^p - norm$.

2.2.2 Unknowns and parameters of the microscopic model

The unknowns in the microscopic model are:

$$\begin{aligned}
\text{Concentration of soil particles in water/soil mixture } C_s^\epsilon &: \quad F \times \Lambda_f^\epsilon \to \mathbb{R}\\
\text{Concentration of deposited soil particles } S_d^\epsilon &: \quad F \times \Lambda_f^\epsilon \to \mathbb{R}\\
\text{Flow velocity } u^\epsilon &: \quad F \times \Lambda_f^\epsilon \to \mathbb{R}\\
\text{Flow pressure } p^\epsilon &: \quad F \times \Lambda_f^\epsilon \to \mathbb{R}\\
\text{Fraction of non-clogging conduit } f_{nc}^\epsilon &: \quad F \times \Lambda_f^\epsilon \to \mathbb{R}
\end{aligned}$$

The parameters in the model are:

$$\begin{array}{rcl} \text{Diffusion Coefficient } D^{\epsilon} & : & F \times \Lambda_f^{\epsilon} \to \mathbb{R} \\ \text{Molecular viscosity } \mu^{\epsilon} & : & F \times \Lambda_f^{\epsilon} \to \mathbb{R} \\ \text{Euler Number } E_u^{\epsilon} & : & F \times \Lambda_f^{\epsilon} \to \mathbb{R} \\ \text{Attachment efficiency for non-clogging conduit } \beta_{nc} & \in & (0,\infty) \\ \text{Attachment efficiency for clogging conduit } \beta_{cl} & \in & (0,\infty) \\ \text{Average overall mass transfer coefficient } \Psi & \in & (0,\infty) \end{array}$$

As a working assumption we assume the following:

$$D^{\epsilon}, \mu^{\epsilon}, E_u^{\epsilon} \in L^{\infty}(F \times \Lambda)$$
$$D^{\epsilon} \text{ and } \mu^{\epsilon} \text{are positive definate.}$$

We thus have the dimensionless coupled equations at the microscopic level for the piping flow erosion as

$$\nabla . u^{\epsilon} = 0 \ \ in \ \ \Lambda^{\epsilon} \tag{2.1}$$

$$[\rho(u^{\epsilon \partial \Lambda_s} - u^{\epsilon}).n] = 0 \quad on \ \ \partial\Lambda_s^{\epsilon} \tag{2.2}$$

$$\frac{\partial}{\partial t}(C_s^{\epsilon} + S_d^{\epsilon}) + \nabla.(u^{\epsilon} C_s^{\epsilon}) = \nabla.(D(t,y)\nabla C_s^{\epsilon}) \quad in \ \ \Lambda_f^{\epsilon} \tag{2.3}$$

$$[C_s^{\epsilon}(u^{\epsilon \partial \Lambda_s} - u^{\epsilon}) + D(t,y)\nabla C_s^{\epsilon}].n = 0 \quad on \ \ \partial\Lambda_s^{\epsilon} \tag{2.4}$$

$$\frac{\partial}{\partial t}(u^{\epsilon}) + (u^{\epsilon}.\nabla)u^{\epsilon} = -E_u(t,y)\nabla p^{\epsilon} + \nabla.(2\mu(t,y)\nabla u^{\epsilon}) \quad in \ \ \Lambda_f^{\epsilon} \tag{2.5}$$

$$[u^{\epsilon}(u^{\epsilon \partial \Lambda_s} - u^{\epsilon}) - E_u(t,y)p^{\epsilon} + 2\mu(t,y)\nabla u^{\epsilon}].n = 0 \quad on \ \ \partial\Lambda_s^{\epsilon} \tag{2.6}$$

$$\frac{\partial S_d^{\epsilon}}{\partial t} = (\beta_{nc} f_{nc}^{\epsilon} + \beta_{cl}(1 - f_{nc}^{\epsilon}))\Psi C_s^{\epsilon} \ \ in \ \ \Lambda_f^{\epsilon} \tag{2.7}$$

$$\frac{\partial f_{nc}^{\epsilon}}{\partial t} + \beta_{nc}\Psi f_{nc}^{\epsilon} C_s^{\epsilon} = 0 \ \ in \ \ \Lambda_s^{\epsilon} \tag{2.8}$$

2.3 The weak formulation of the microscopic problem

On the weak formulation of the microscopic problem we postulate definition 2.1

Definition 2.1. The functions

$$\begin{array}{rcl} C_s^{\epsilon} & \in & H^1(F; L^2(\Lambda_f^{\epsilon})) \\ u^{\epsilon} & \in & H^1(F; L^2(\Lambda_f^{\epsilon})) \\ p^{\epsilon} & \in & W^{1,\infty}(F) \\ S_d^{\epsilon} & \in & W^{1,\infty}(F) \\ f_{nc}^{\epsilon} & \in & W^{1,\infty}(F) \end{array}$$

are called weak solutions to (2.1)- (2.8) if for every $t \in F$ the following holds

$$\langle \frac{\partial}{\partial t}(C_s^{\epsilon} + S_d^{\epsilon}), \omega \rangle_{L^2(\Lambda_f^{\epsilon})} + \langle D\nabla C_s^{\epsilon}, \nabla\omega \rangle_{L^2(\Lambda_f^{\epsilon})} = \langle u^{\epsilon} C_s^{\epsilon}, \nabla\omega \rangle_{L^2(\Lambda_f^{\epsilon})} \tag{2.9}$$

$$\langle \frac{\partial}{\partial t} u^\epsilon, \Pi \rangle_{L^2(\Lambda_f^\epsilon)} + \langle 2\mu \nabla u^\epsilon, \nabla \Pi \rangle_{L^2(\Lambda_f^\epsilon)} = \langle u^\epsilon u^\epsilon, \nabla \Pi \rangle_{L^2(\Lambda_f^\epsilon)} + \langle E_u p^\epsilon, \nabla \Pi \rangle_{L^2(\Lambda_f^\epsilon)} \quad (2.10)$$

for test functions $\omega, \Pi \in H^1(F \times \Lambda^\epsilon)$ such that $\omega_{|\partial \Lambda_s^\epsilon} = 0$ and $\Pi_{|\partial \Lambda_s^\epsilon} = 0$.

with

$$\frac{\partial S_d^\epsilon}{\partial t} = (\beta_{nc} f_{nc}^\epsilon + \beta_{cl}(1 - f_{nc}^\epsilon))\Psi C_s^\epsilon$$

$$\frac{\partial f_{nc}^\epsilon}{\partial t} + \beta_{nc} \Psi f_{nc}^\epsilon C_s^\epsilon = 0$$

2.4 The $\epsilon-$independent prior estimate (Energy Estimates)

In this section and the next, the well-posedness of the microscopic problem is discussed. We prove the existence of solution to (2.9) and (2.10) thus postulate Lemma 2.1.

Lemma 2.1. *There exist constants k_1 and k_2 which are independent of ϵ such that*

$$||u^\epsilon||_{L^2(F;H^1(\Lambda_f^\epsilon))} + ||\nabla_t u^\epsilon||_{L^2(F;L^2(\Lambda_f^\epsilon))} \leq k_1, \quad (2.11)$$

$$||C_s^\epsilon||_{L^2(F;H^1(\Lambda_f^\epsilon))} + ||\nabla_t C_s^\epsilon||_{L^2(F;L^2(\Lambda_f^\epsilon))} \leq k_2. \quad (2.12)$$

Proof. *of* (2.11)

From (2.10) we choose the test function $\Pi = u^\epsilon$

$$\langle \frac{\partial}{\partial t} u^\epsilon, u^\epsilon \rangle + \langle 2\mu \nabla u^\epsilon, \nabla u^\epsilon \rangle = \langle u^\epsilon u^\epsilon, \nabla u^\epsilon \rangle + \langle E_u p^\epsilon, \nabla u^\epsilon \rangle$$

let $2\mu = g_0 > 0$, $E_u = e_0 > 0$ for $g_0, e_0 \in \mathbb{R}$ and applying Cauchy - Schwarz [13] we get

$$\frac{1}{2}\frac{\partial}{\partial t}||u^\epsilon||^2 + g_0||\nabla u^\epsilon||^2 \leq ||u^\epsilon u^\epsilon|| ||\nabla u^\epsilon|| + e_0||p^\epsilon|| ||\nabla u^\epsilon||$$

Using Young's inequality (variant) [14] we obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}||u^\epsilon||^2 + g_0||\nabla u^\epsilon||^2 \leq \delta_1||\nabla u^\epsilon||^2 + c(\delta_1)||u^\epsilon||^4 + \delta_2||\nabla u^\epsilon||^2 + c(\delta_2)e_0||p^\epsilon||^2$$

for $\delta_1, \delta_2 > 0$, $c(\delta_1) = \dfrac{1}{4\delta_1}$ and $c(\delta_2) = \dfrac{1}{4\delta_2}$
grouping similar terms gives

$$\frac{1}{2}\frac{\partial}{\partial t}||u^\epsilon||^2 + (g_0 - \delta_1 - \delta_2)||\nabla u^\epsilon||^2 \leq c(\delta_1)||u^\epsilon||^4 + c(\delta_2)e_0||p^\epsilon||^2$$

by Friedrich's inequality [15], $||u^\epsilon||^2$ can be bounded by its weak derivative thus for $\delta_1 + \delta_2 \in (0, g_0)$ we can write

$$||\nabla u^\epsilon||^2 \leq \frac{1}{(g_0 - \delta_1 - \delta_2)}\left(c(\delta_1)||u^\epsilon||^4 + c(\delta_2)e_0||p^\epsilon||^2_{L^\infty} - \frac{1}{2}\frac{\partial}{\partial t}||u^\epsilon||^2\right)$$

hence by integrating we have

$$\begin{aligned}\int_0^t ||\nabla u^\epsilon||^2 dt^* &\leq \frac{1}{(g_0 - \delta_1 - \delta_2)}\left(\left(\int_0^t (c(\delta_1)||u^\epsilon||^4 + c(\delta_2)e_0||p^\epsilon||^2_{L^\infty})dt^*\right) - \frac{1}{2}|||u^\epsilon||^2\right) \\ &= k_1 \\ &< \infty \textit{ since } ||u^\epsilon||, ||p^\epsilon|| \textit{ are bounded.}\end{aligned}$$

□

Proof. of (2.12)

From (2.9) choose $\omega = C_s^\epsilon$

$$\langle \frac{\partial}{\partial t}(C_s^\epsilon + S_d^\epsilon), C_s^\epsilon \rangle + \langle D\nabla C_s^\epsilon, \nabla C_s^\epsilon \rangle = \langle u_\epsilon C_s^\epsilon, \nabla C_s^\epsilon \rangle$$

$$\langle \frac{\partial}{\partial t} C_s^\epsilon, C_s^\epsilon \rangle + \langle \frac{\partial}{\partial t} S_d^\epsilon, C_s^\epsilon \rangle + \langle D\nabla C_s^\epsilon, \nabla C_s^\epsilon \rangle = \langle u_\epsilon C_s^\epsilon, \nabla C_s^\epsilon \rangle$$

$$\langle \frac{\partial}{\partial t} C_s^\epsilon, C_s^\epsilon \rangle + \langle D\nabla C_s^\epsilon, \nabla C_s^\epsilon \rangle = \langle u_\epsilon C_s^\epsilon, \nabla C_s^\epsilon \rangle - \langle \frac{\partial}{\partial t} S_d^\epsilon, C_s^\epsilon \rangle$$

For $D = d_0 > 0$ and applying Cauchy-Schwarz we get

$$\frac{1}{2}\frac{\partial}{\partial t}||C_s^\epsilon||^2 + d_0||\nabla C_s^\epsilon||^2 \leq ||u^\epsilon C_s^\epsilon|| ||\nabla C_s^\epsilon|| + ||\frac{\partial}{\partial t} S_d^\epsilon|| ||C_s^\epsilon||$$

Next we apply the Arithmetic mean-geometric mean inequality [13] and Variant Young's inequality and obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}||C_s^\epsilon||^2 + d_0||\nabla C_s^\epsilon||^2 \leq \delta||\nabla C_s^\epsilon||^2 + Q(\delta)||u^\epsilon||^2||C_s^\epsilon||^2 + \frac{1}{2}||\frac{\partial S_d^\epsilon}{\partial t}||^2 + \frac{1}{2}||C_s^\epsilon||^2$$

for $\delta > 0$ and $Q(\delta) = \frac{1}{4\delta}$.
this inequality can also be grouped as

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}||C_s^\epsilon||^2 + (d_0 - \delta)||\nabla C_s^\epsilon||^2 &\leq \frac{1}{2}||\frac{\partial}{\partial t} S_d^\epsilon||^2 \\ &+ \left(\frac{1}{2} + Q(\delta)||u^\epsilon||^2\right)||C_s^\epsilon||^2 \end{aligned} \tag{2.13}$$

by Gronwall's inequality [16] for $\delta \in (0, d_0)$, $d_0 > 0$, we can write

$$\begin{aligned} ||C_s^\epsilon||^2 &\leq e^{\int_0^t (1+2Q(\delta)||u^\epsilon||^2)dt^*} \left(||C_s^\epsilon(0,x)||^2 + \int_0^t ||\frac{\partial}{\partial t} S_d^\epsilon||_{L_\infty}^2 dt^*\right) \\ &= e^{(t+2Q(\delta)\int_0^t ||u^\epsilon||^2 dt^*)} \left(||C_s^\epsilon(0,x)||^2 + \int_0^t ||\frac{\partial}{\partial t} S_d^\epsilon||_{L_\infty}^2 dt^*\right) \\ &\leq e^{(T+2Q(\delta)\int_0^T ||u^\epsilon||^2 dt^*)} \left(||C_s^\epsilon(0,x)||^2 + \int_0^T ||\frac{\partial}{\partial t} S_d^\epsilon||_{L_\infty}^2 dt^*\right) \end{aligned}$$

Hence C_s^ϵ is bounded,
from (2.13) we have

$$||\nabla C_s^\epsilon||^2 \leq \frac{1}{2(d_0 - \delta)}\left((1 + Q(\delta)||u^\epsilon||^2)||C_s^\epsilon||^2 + ||\frac{\partial}{\partial t} S_d^\epsilon||_{L_\infty}^2 - \frac{\partial}{\partial t}||C_s^\epsilon||^2\right)$$

Integrate on $(0, t)$

$$\begin{aligned} \int_0^t ||\nabla C_s^\epsilon||^2 dt^* &\leq \frac{1}{2(d_0 - \delta)}\left(\int_0^t (1 + Q(\delta)||u^\epsilon||^2)||C_s^\epsilon||^2 dt^*\right. \\ &+ \left.\int_0^t ||\frac{\partial}{\partial t} S_d^\epsilon||_{L_\infty}^2 dt^* - ||C_s^\epsilon||^2\right) \\ &= k_2 < \infty \end{aligned}$$

□

2.5 Compactness results (Two-scale)

From the boundedness results obtained and *theorem* 1.3, we have the following results: there exist $u_0, C_{s0} \in L^2(F \times \Lambda \times Y)$, $p_0, S_{d0} \in L^\infty(F \times \Lambda \times Y)$, $C_{s1}, u_1 \in L^2(F \times \Lambda; W)$ such that

(i) $u^\epsilon \rightharpoonup^2 u_0$, $C_s^\epsilon \rightharpoonup^2 C_{s0}$, $p^\epsilon \rightharpoonup^2 p_0$ and $S_d^\epsilon \rightharpoonup^2 S_{d0}$

(ii) $\nabla u^\epsilon \rightharpoonup^2 \nabla_x u_0 + \nabla_y u_1$ and $\nabla C_s^\epsilon \rightharpoonup^2 \nabla_x C_{s0} + \nabla_y C_{s1}$

(iii) $u^\epsilon \rightharpoonup \hat{u_0}(t,x,y)$ weakly, $p^\epsilon \rightharpoonup \hat{p_0}(t,x,y)$ weakly, $C_s^\epsilon \rightharpoonup \hat{C_{s0}}(t,x,y)$ weakly and

(iv) $u^\epsilon \rightharpoonup \tilde{u_0}(t,x)$ strongly, $p^\epsilon \rightharpoonup \tilde{p_0}(t,x)$ strongly, $C_s^\epsilon \rightharpoonup \tilde{C_{s0}}(t,x)$ strongly

consequently uniqueness implies

(v) $\tilde{u_0}(t,x) = \hat{u_0}(t,x,y) = u_0(t,x)$

(vi) $\tilde{C_{s0}}(t,x) = \hat{C_{s0}}(t,x,y) = C_{s0}(t,x)$

(vii) $\tilde{p_0}(t,x) = \hat{p_0}(t,x,y) = p_0(t,x)$

2.6 Passing the microscopic problem to $\epsilon \to 0$ via two scale convergence

From (2.10) we have

$$\int_F \int_\Lambda \nabla_t u^\epsilon \Pi dxdt + \int_F \int_\Lambda 2\mu(t, \frac{x}{\epsilon}) \nabla u^\epsilon \nabla \Pi dxdt = \int_F \int_\Lambda u^\epsilon u^\epsilon \nabla \Pi dxdt +$$
$$\int_F \int_\Lambda E_u(t, \frac{x}{\epsilon}) p^\epsilon \nabla \Pi dxdt$$

for all $\Pi \in H^1(F \times \Lambda)$. We choose the test function $\Pi(t,x) = \Pi_0(t,x) + \epsilon \Pi_1(t,x,\frac{x}{\epsilon})$ where $\Pi_0, \Pi_1 \in C_0^\infty(F \times \Lambda) \times C_0^\infty(F \times \Lambda; C_\#^\infty(Y))$ and obtain:

$$\int_F \int_\Lambda \nabla_t u^\epsilon (\Pi_0 + \epsilon \Pi_1) dxdt + \int_F \int_\Lambda 2\mu(t, \frac{x}{\epsilon}) \nabla u^\epsilon \nabla (\Pi_0 + \epsilon \Pi_1) dxdt =$$
$$\int_F \int_\Lambda u^\epsilon u^\epsilon \nabla (\Pi_0 + \epsilon \Pi_1) dxdt + \int_F \int_\Lambda E_u(t, \frac{x}{\epsilon}) p^\epsilon \nabla (\Pi_0 + \epsilon \Pi_1) dxdt$$

this expands to

$$\int_F \int_\Lambda \nabla_t u^\epsilon (\Pi_0 + \epsilon \Pi_1) dxdt + \int_F \int_\Lambda 2\mu(t, \frac{x}{\epsilon}) \nabla u^\epsilon (\nabla_x \Pi_0 + \epsilon \nabla_x \Pi_1 + \nabla_y \Pi_1) dxdt =$$
$$\int_F \int_\Lambda u^\epsilon u^\epsilon (\nabla_x \Pi_0 + \epsilon \nabla_x \Pi_1 + \nabla_y \Pi_1) dxdt + \int_F \int_\Lambda E_u(t, \frac{x}{\epsilon}) p^\epsilon (\nabla_x \Pi_0 + \epsilon \nabla_x \Pi_1 + \nabla_y \Pi_1) dxdt$$

grouping we have

$$\int_F \int_\Lambda \nabla_t u^\epsilon \Pi_0 dxdt + \int_F \int_\Lambda 2\mu(x, \frac{x}{\epsilon}) \nabla u^\epsilon (\nabla_x \Pi_0 + \nabla_y \Pi_1) dxdt + \epsilon \int_F \int_\Lambda \nabla u^\epsilon \Pi_1 dxdt +$$
$$\epsilon \int_F \int_\Lambda 2\mu(t, \frac{x}{\epsilon}) \nabla u^\epsilon \nabla_x \Pi_1 dxdt = \int_F \int_\Lambda u^\epsilon u^\epsilon (\nabla_x \Pi_0 + \nabla_y \Pi_1) dxdt +$$
$$\int_F \int_\Lambda E_u(t, \frac{x}{\epsilon}) p^\epsilon (\nabla_x \Pi_0 + \nabla_y \Pi_1) dxdt + \epsilon \int_F \int_\Lambda u^\epsilon u^\epsilon \nabla_x \Pi_1 dxdt +$$
$$\epsilon \int_F \int_\Lambda E_u(t, \frac{x}{\epsilon}) p^\epsilon \nabla_x \Pi_1 dxdt$$

Passing $\epsilon \to 0$ with

$$\nabla u^{\epsilon} \rightharpoonup^{2} \nabla_x u_0(t,x) + \nabla_y u_1(t,x,y)$$
$$u^{\epsilon} \rightharpoonup^{2} u_0(t,x), \; and \; p^{\epsilon} \rightharpoonup^{2} p_0(t,x)$$

we obtain

$$\frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf} \nabla_t u_0 \Pi_0 dydxdt + \frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf} 2\mu(x,y)(\nabla_x u_0 + \nabla_y u_1)(\nabla_x \Pi_0 + \nabla_y \Pi_1)dydxdt = \frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf} u_0 u_0(\nabla_x \Pi_0 + \nabla_y \Pi_1)dydxdt + \frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf} E_u(t,y)p_0(\nabla_x \Pi_0 + \nabla_y \Pi_1)dydxdt$$

which is the weak form of the limit two scale problem. Next we expand and shift the derivatives from the test functions:

$$\frac{1}{|Y|}\int_F\int_\Lambda\left(\int_{Yf} \nabla_t u_0 dy\right)\Pi_0 dxdt - \frac{1}{|Y|}\int_F\int_\Lambda \nabla_x.\left(\int_{Yf} 2\mu(t,y)(\nabla_x u_0 + \nabla_y u_1)dy\right)\Pi_0 dxdt - \frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf}\nabla_y.\left(2\mu(t,y)(\nabla_x u_0 + \nabla_y u_1)\right)\Pi_1 dydxdt = -\frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf}\nabla_x.(u_0u_0)\Pi_0 dydxdt - \frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf}\nabla_y.(u_0u_0)\Pi_1 dydxdt - \frac{1}{|Y|}\int_F\int_\Lambda \nabla_x.\left(\int_{Yf} E_u(t,y)dy\right)p_0\Pi_0 dxdt - \frac{1}{|Y|}\int_F\int_\Lambda\int_{Yf}\nabla_y.\left(E_u(t,y)p_0\right)\Pi_1 dydxdt$$

Now we let $\Pi_0 = 0, \Pi_1 = 0$ within $F \times \Lambda$, $F \times \Lambda \times Y$ and their boundaries we obtain the homogenized equation:

$$\phi\frac{\partial u_0}{\partial t} - \frac{1}{|Y|}\nabla_x.\left(\int_{Yf} 2\mu(t,y)(\nabla_x u_0 + \nabla_y u_1)dy\right) + \phi\nabla_x.(u_0u_0) + \bar{E}_u\nabla_x p_0 = 0 \tag{2.14}$$

almost everywhere in $F \times \Lambda$ with $\bar{E}_u = \frac{1}{|Y|}\int_{Yf} E_u(t,y)dy$, $\phi = \frac{|Y^f|}{|Y|}$ and

$$-\nabla_y.\left(2\mu(t,y)(\nabla_x u_0 + \nabla_y u_1)\right) = 0 \tag{2.15}$$

almost everywhere in $F \times \Lambda \times Y$.

It must be noted that (2.14) and (2.15) are respectively the macro problem and cell problem.

Similarly from the weak formulation in (2.9) we have

$$\int_F\int_\Lambda \nabla_t(C_s^\epsilon + S_d^\epsilon)\omega dxdt + \int_F\int_\Lambda D(t,\frac{x}{\epsilon})\nabla C_s^\epsilon \nabla\omega dxdt = \int_F\int_\omega u^\epsilon C_s^\epsilon \nabla\omega dxdt$$

for all $\omega \in H^1(F \times \Lambda)$.

We choose test function

$$\omega(t,x) = \omega_0(t,x) + \epsilon\omega_1(t,x,\frac{x}{\epsilon})$$

where $(\omega_0, \omega_1) \in C_0^\infty(F \times \Lambda) \times C_0^\infty(F \times \Lambda; C_\#^\infty(Y))$ and obtain:

$$\begin{aligned}\int_F\int_\Lambda \nabla_t(C_s^\epsilon + S_d^\epsilon)(\omega_0 + \epsilon\omega_1)dxdt + \int_F\int_\Lambda D(t,\frac{x}{\epsilon})\nabla C_s^\epsilon \nabla(\omega_0 + \epsilon\omega_1)dxdt &= \\ \int_F\int_\Lambda u^\epsilon C_s^\epsilon \nabla(\omega_0 + \epsilon\omega_1)dxdt\end{aligned}$$

which simplifies to

$$\begin{aligned}\int_F\int_\Lambda \nabla_t(C_s^\epsilon + S_d^\epsilon)(\omega_0 + \epsilon\omega_1)dxdt + \\ \int_F\int_\Lambda D(t,\frac{x}{\epsilon})\nabla C_s^\epsilon(\nabla_x\omega_0 + \epsilon\nabla_x\omega_1 + \nabla_y\omega_1)dxdt &= \\ \int_F\int_\Lambda u^\epsilon C_s^\epsilon(\nabla_x\omega_0 + \epsilon\nabla_x\omega_1 + \nabla_y\omega_1)dxdt\end{aligned}$$

this further expands to

$$\begin{aligned}\int_F\int_\Lambda \nabla_t(C_s^\epsilon + S_d^\epsilon)\omega_0 dxdt + \int_F\int_\Lambda D(t,\frac{x}{\epsilon})\nabla C_s^\epsilon(\nabla_x\omega_0 + \nabla_y\omega_1)dxdt + \\ \epsilon\int_F\int_\Lambda \nabla_t(C_s^\epsilon + S_d^\epsilon)\omega_1 dxdt + \epsilon\int_F\int_\Lambda D(t,\frac{x}{\epsilon})\nabla C_s^\epsilon\nabla_x\omega_1 dxdt + &= \\ \int_F\int_\Lambda u^\epsilon C_s^\epsilon(\nabla_x\omega_0 + +\nabla_y\omega_1)dxdt + \epsilon\int_F\int_\Lambda u^\epsilon C_s^\epsilon\nabla_x\omega_1 dxdt\end{aligned}$$

Passing $\epsilon \to 0$ with

$$\nabla C_s^\epsilon \rightharpoonup^2 \nabla_x C_{s0}(t,x) + \nabla_y C_{s1}(t,x,y)$$

$$C_s^\epsilon \rightharpoonup^2 C_{s0}(t,x), S_d^\epsilon \rightharpoonup^2 S_{d0}(t,x)$$

$$u^\epsilon \rightharpoonup^2 u_0(t,x)$$

we obtain

$$\begin{aligned}\frac{1}{|Y|}\int_F\int_\Lambda\int_{Y_f} \nabla_t(C_{s0} + S_{d0})\omega_0 dydxdt + \\ \frac{1}{|Y|}\int_F\int_\Lambda\int_{Y_f} D(t,y)(\nabla_x C_{s0} + \nabla_y C_{s1})(\nabla_x\omega_0 + \nabla_y\omega_1)dydxdt &= \\ \frac{1}{|Y|}\int_F\int_\Lambda\int_{Y_f} u_0 C_{s0}(\nabla_x\omega_0 + \nabla_y\omega_1)dydxdt\end{aligned}$$

Next we expand and shift the derivatives from the test functions

$$\frac{1}{|Y|}\int_F\int_\Lambda\left(\int_{Y_f}\nabla_t(C_{s0}+S_{d0})dy\right)\omega_0 dxdt -$$
$$\frac{1}{|Y|}\int_F\int_\Lambda\nabla_x.\left(\int_{Y_f}D(t,y)(\nabla_x C_{s0}+\nabla_y C_{s1})dy\right)\omega_0 dxdt -$$
$$\frac{1}{|Y|}\int_F\int_{\Lambda\times Y_f}\nabla_y.\left(D(t,y)(\nabla_x C_{s0}+\nabla_y C_{s1})\right)\omega_1 dydxdt \quad =$$
$$-\frac{1}{|Y|}\int_F\int_{\Lambda\times Y_f}\nabla_x.(u_0 C_{s0})\omega_0 dydxdt$$
$$-\frac{1}{|Y|}\int_F\int_{\Lambda\times Y_f}\nabla_y.(u_0 C_{s0})\omega_1 dydxdt$$

Choosing $\omega_0 = 0, \omega_1 = 0$ within $F\times\Lambda$, $F\times\Lambda\times Y$ and their boundaries we get

$$\frac{|Y_f|}{|Y|}\nabla_t(C_{s0}+S_{d0}) - \frac{1}{|Y|}\nabla_x.\left(\int_{Y_f}D(t,y)(\nabla_x C_{s0}+\nabla_y C_{s1})dy\right) +$$
$$\frac{|Y_f|}{|Y|}\nabla_x.(u_0 C_{s0}) = 0 \;\; in \;\; F\times\Lambda \tag{2.16}$$

$$\nabla_y.\left(D(t,y)(\nabla_x C_{s0}+\nabla_y C_{s1})\right) = 0 \;\; in \;\; F\times\Lambda\times Y \tag{2.17}$$

Equations (2.16) and (2.17) coincides with the equations obtained through the formal homogenization, these are respectively the homogenized macroscopic model and the equation at the unit cell level from which the cell problem can be obtained. We have thus shown that the results obtained from the formal homogenization can be verified through this rigorous homogenization process via two scale convergence.

3 Numerical Computation of the Homogenized Macroscopic Problem

The model is used to simulate the piping flow erosion with deposition in a highly erodable soil under a tangential flow instigated by seepage of water through the embarkment. We imposed a constant pressure drop between inlet and outlet with a constant flux at the inlet, the tangential velocities are assumed continuous across $\partial\Lambda$. The cell problems and macroscopic equations were discretize using the finite element method (Galerkin).The momentum equation was decoupled using the Incremental Pressure Correction Scheme (IPCS) [17, 18].

The numerical task performed include:

(i) Computing the cell problem:

$$-\frac{\partial}{\partial y_i}\left(D_{ik}(t,y)\frac{\partial\chi_j^c}{\partial y_k}\right) = \frac{\partial}{\partial y_i}D_{ij}(t,y)$$

$$B.C: \quad \left(D_{ij}(t,y) + D_{ik}(t,y)\frac{\partial\chi_j^c}{\partial y_k}\right).n(y) = 0$$

$$-\frac{\partial}{\partial y_i}\left(\mu_{ik}(t,y)\frac{\partial \chi_j^{\mu}}{\partial y_k}\right) = \frac{\partial}{\partial y_i}\mu_{ij}(t,y)$$

$$B.C: \quad \left(\mu_{ij}(t,y) + \mu_{ik}(t,y)\frac{\partial \chi_j^{c}}{\partial y_k}\right).n(y) = 0$$

for $i,k = 1,2$,$j = 1,2$ and $\chi_j^c, \chi_j^{\mu} - Y$ periodic

(ii) Using the results in (i) to compute the Homogenized effective characteristic coefficients:

$$D^h_= \frac{1}{|Y|}\int_{Y_f}\left(D_{ij}(t,y) + D_{ik}(t,y)\frac{\partial \chi_j^c}{\partial y_k}\right)dy$$

$$\mu^h_= \frac{1}{|Y|}\int_{Y_f}\left(\mu_{ij}(t,y) + \mu_{ik}(t,y)\frac{\partial \chi_j^{\mu}}{\partial y_k}\right)dy$$

$$\bar{E_u} = \frac{1}{|Y|}\int_{Y_f} E_u dy$$

for $i,k = 1,2$,$j = 1,2$

(iii) We then solved 2D the Homogenized problem :

$$\phi\frac{\partial u_{0_i}}{\partial t} + \phi u_{0_j}\frac{\partial u_{0_i}}{\partial x_j} = -\bar{E}_u\frac{\partial p_0}{\partial x_i} + 2\mu^h\frac{\partial^2 u_{0_i}}{\partial x_i^2}$$

$$\phi\frac{\partial C_{s0}}{\partial t} + \phi\frac{\partial S_{d0}}{\partial t} + \phi u_{0_j}\frac{\partial C_{s0}}{\partial x_j} = D^h\frac{\partial^2 C_{s0}}{\partial x_i^2}$$

for $i = 1,2$,$j = 1,2$

With concluding equations for deposition and pore space dynamics

$$\frac{\partial S_{d0}}{\partial t} = [\beta_{nc}f_{nc_0} + \beta_{cl}(1 - f_{nc_0})]\Psi C_{s0}$$

$$\frac{\partial f_{nc_0}}{\partial t} + \beta_{nc}\Psi f_{nc_0}C_{s0} = 0$$

Initial and boundary conditions imposed are:

$$f_{nc_0}(0,x_1,x_2) = 1, \quad S_{d0}(0,x_1,x_2) = 0$$
$$C_{s0}(t,0,x_2) = 0, \quad S_{d0}(t,0,x_2) = 0$$
$$p_0(t,0,x_2) = Pin, \quad p_0(t,L,x_2) = 0$$
$$u_0(t,x_1,0) = 0, \quad u_0(t,x_1,L) = 0$$

Numerical values of the parameters used are listed in Table 1 for a rectangular soil domain of dimensions $2m \times 1m$ with compacity $C_{soil} = 1 - \phi$. The characteristic erosion time was computed as in [19], $t_{er} = \dfrac{2L\rho^p}{k_{er}p_{in}} = 3000hrs$. The density of the water/soil mixture $\rho = \phi(\rho^p - \rho^w) + \rho^w$, where ρ^w and ρ^p are water and soil particles density. The time-stepping was performed using Backward Euler Method [20] with a time step $\delta t = 0.2$ for $0 \leq t \leq 20$

Table 1. Numerical values of parameters

L	H	p_{in}	ρ^w	ρ^p	ϕ
2m	1m	0.1	$1000kgm^{-3}$	$2700kgm^{-3}$	0.35
k_{er}	Ψ	β_{nc}	β_{cl}	E_u	C_{soil}
$0.01\ sm^{-1}$	$1.22 \times 10^{-6} ms^{-1}$	0.73	3.96×10^{-4}	0.00424	0.65

4 Numerical Results and Discussion

Using the Galerkin finite element on an unstructured triangular mesh we divided the soil domain into 10116 elements as in Fig. 2. Define an inflow at $x_1 = 0$ and an outflow at $x_1 = 2$. No slip conditions are imposed on the walls $x_2 = 0$ and $x_2 = 1$. The soil domain was subjected to a constant pressure drop of 0.1. Under a parabolic velocity profile realized from the momentum equation (Fig. 3), the results obtained in terms of percentage increase and decrease in soil particle deposition and entrainment are presented.

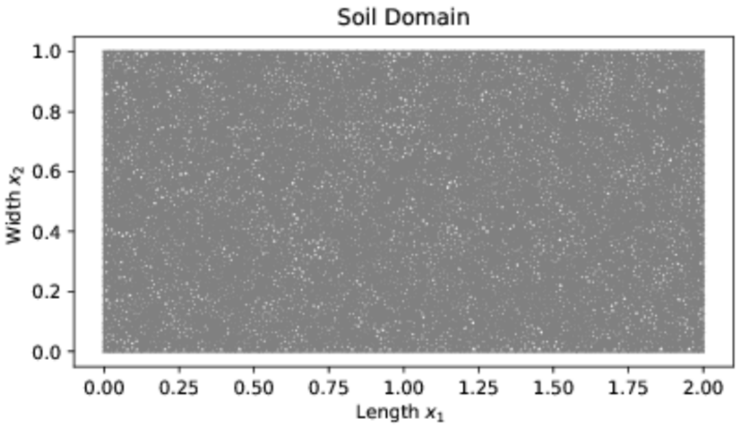

Fig. 2. Meshing of the soil domain

We observed a decrease in concentration of soil particles in the flow over the period simulated, to substantiate the extent of decrease we computed the percentage decrease in soil particle concentration in the flow. At the early stages of the simulation, eroded soil particles transforms the flow into a concentrated suspension causing enlargement of the pipe downstream. However these eroded soil particles are entrained towards the outflow, variations in levels of percentage decrease in concentration of particles in the flow as depicted by our model can be seen in Fig. 4. A rapid decrease in soil particle concentration was observed at the early stages of the process with higher levels (almost 50%) close to inflow, this is due to the higher pressure levels at the inflow thereby propagating the eroded particles from inflow towards outflow. It can be seen that, at the later stages of the process the decrease in soil particle is much slower as most of the particles entrained are deposited at the outlet.

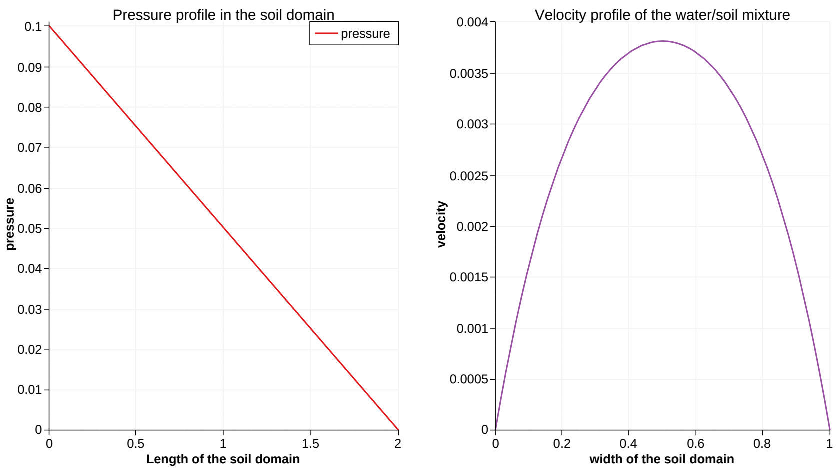

Fig. 3. (a) A snapshot of the Pressure profile in the soil domain (b) Velocity profile in the soil domain

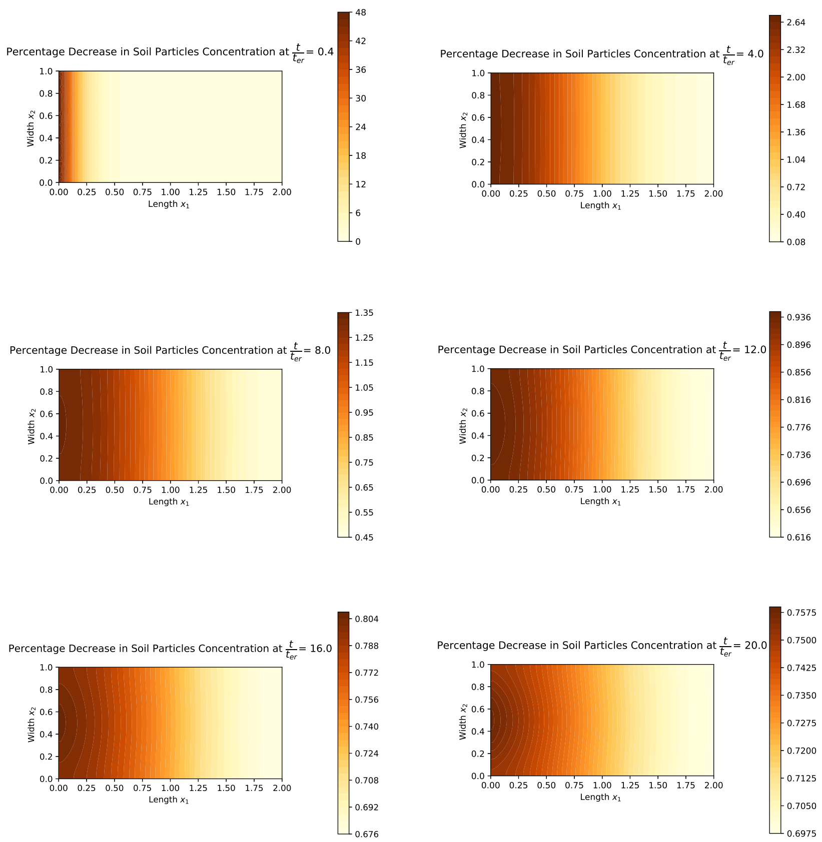

Fig. 4. Percentage decrease in soil concentration in the water/soil mixture

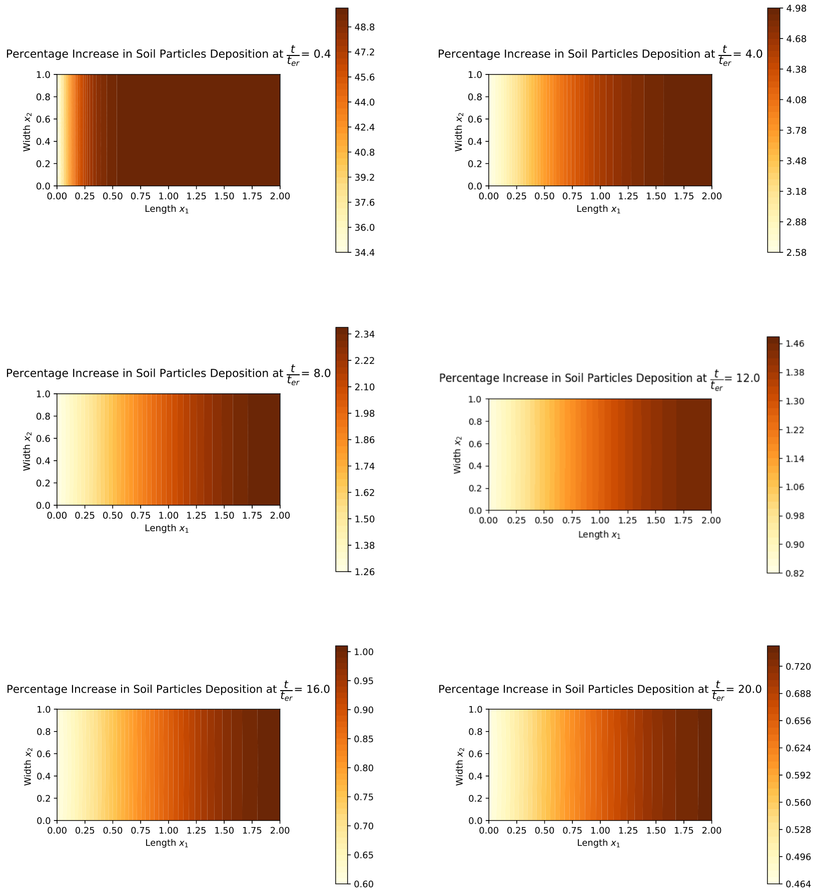

Fig. 5. Percentage increase in concentration of soil particles deposited

Deposition of soil particles entrained occurred over the period. We observed further from the percentage increase in deposition computed that, the concentrated suspension at the early stages of the piping flow erosion causes a rapid increase in soil deposition. Though deposition increases, however we observed as in Fig. 5 that this increase is slower with time thus deposition increases at a decreasing rate.

5 Conclusion

The paper started off with a microscopic system of equations for modelling piping flow erosion with deposition in a spatially heterogeneous soil with periodic positioning of pores. The main aim is to provide a rigorous backing for the formal homogenization results obtained in the earlier paper, thus we proved the energy bound of the weak solution to the microscopic model thereby preparing the way for the rigorous passing of $\epsilon \to 0$ via two scale convergence. We have accomplished the task of rigorous homogenization of the microscopic model and have obtained a homogenized macroscopic model with effective coefficients capable of simulating the piping flow erosion phenomena in a spatially heterogeneous soil. A strong observation from the numerical simulation was that, soil particle concentration in the water/soil mixture decreases but at a decreasing rate whereas soil particle deposition increases at regions with increasing amount of particle concentration in the flow causing a reduction in bare pore spaces across the soil domain. These numerical results from the proposed model clearly shows the various trends [21, 22] associated with soil particle concentration in the flow and deposition in the piping phenomena.

Competing Interests

Authors have declared that no competing interests exist.

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