# The Decomposition for the $\alpha$-level Fuzzy Soft Sets

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## Abstract

In this paper, the properties of $\alpha$-level fuzzy soft sets and $\alpha$-level fuzzy soft lattices are discussed. Firstly, some soft operations between $\alpha$-level fuzzy soft sets (lattices) are defined, such as the soft union and intersection operations, and illustrates them by the examples. Secondly, the relation between the $\alpha$-level fuzzy soft lattices and the $\alpha$-level fuzzy soft sets are studied, we mainly verify the properties not valid in the case discussed, but tenable in the classical soft set theory. Thirdly, the decomposition theorem for the $\alpha$-level fuzzy soft sets is proved. Lastly, a simple application in skin disease diagnosis is illustrated.

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# 1    Introduction

Many scholars are researching the comprehensive application of Soft set [1] [2], fuzzy set [3], formal concept analysis [4] and lattice [5]. For example, in [6], the authors defined fuzzy soft set by putting fuzzy set and soft set together, and based on [6], researchers presented some more properties of fuzzy soft union and fuzzy soft intersection in [7], and authors defined generalized fuzzy soft sets and studied some of their properties and application in [8]. And others have done the study of fuzzy soft lattice from perspective of both fuzzy set and lattice, such as, in [9], the author defined the concept for soft lattices, and in [10], the notion of fuzzy soft lattice was defined and extended the notion of a fuzzy lattice to include the algebraic structures of soft set. To attain more information you can see [11]- [15]. In [16], authors defined the $\alpha$-level fuzzy soft sets (lattices). In this article, we mainly discuss the $\alpha$-level fuzzy soft sets and $\alpha$-level fuzzy soft lattices. The rest of this article is listed as followings. In section 2, some basic notions about fuzzy sets, soft sets, fuzzy soft sets, and soft lattices are reviewed concisely. In section 3, we define the operations for $\alpha$-level fuzzy soft sets (lattices), illustrate them with examples, study their properties, and discuss their relationship. Meanwhile, we point out some properties which can not be hold in present situation, but hold under classical soft sets. In section 4, the decomposition theorem of the $\alpha$-level fuzzy soft sets is studied, and test our result using examples. In section 5, a simple application about diagnosis in skin disease is given which can help us to understand $\alpha$-level fuzzy soft sets. In conclusion, the article is summarized in section 6.

# 2    Basic Knowledge

Suppose the reader is familiar with the knowledge of fuzzy sets, soft sets, orders and lattices. In this part, we only review the most basic contents about them.

**Definition 2.1.**   [1] Let $U$ be an initial universe set and $E$ be a parameters set. Let $\mathcal{P}(U)$ be the power set of $U$, $A \subset E$. Then a pair $(F, A)$ is called a  soft  set over $U$, where $F : A \to \mathcal{P}(U)$ is a mapping.

That is, the soft set is a parameterized family of subsets of the set $U$. Every set $F(\varepsilon), \forall \varepsilon \in E$, from this family may be considered as the set of $\varepsilon$-elements of the soft set $(F, E)$, or considered as the set of $\varepsilon$-approximate elements of the soft set. According to this manner, we can view a soft set $(F, E)$ as consisting of collection of approximations: $(F, E) = \{F(\varepsilon) \mid e \in E\} = \{(F(e), e) \mid \varepsilon \in M\}$.

**Definition 2.2.**   [3]  Let $U$ be a nonempty set which is universe.
(i)   A fuzzy set is a class of objects $U$ with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one.
(ii) A fuzzy subset $\mu$ of $U$ is defined by the membership function $\mu : U \to [0, 1]$. For $x \in U$, the membership value $\mu(x)$ essentially specifies the degree to which $x$ belongs to the fuzzy subset $\mu$.

**Definition 2.3.**   [5]  Let $P$ be a set. An order (partial order) on $P$ is a binary relation $\preceq$ such that, for all $x, y, z \in P$, (i) $x \preceq x$; (ii) $x \preceq y, y \preceq x \Rightarrow x = y$; (iii) $x \preceq y, y \preceq z \Rightarrow x \preceq z$. And$(P, \preceq)$ is a partial order set.

**Definition 2.4.**   [5]  Let $P$ be a non-empty ordered set.
(i)  If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then $P$ is called a lattice;
(ii)  If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then $P$ is called a complete lattice.

**Claim 1.**   *(i) $x \preceq y$ means $x$ is less than $y$, then there is a line between $x$ and $y$, and $x$ is below of $y$ in their diagram.*

*(ii) $x \parallel y$ mean $x$ and $y$ is non-comparability, and no line between them.*
*(iii) In the following, we will use the operators $\vee$ and $\wedge$. For fuzzy set, $x \vee y = max\{x, y\}$ and $x \wedge y = min\{x, y\}$; for lattice, $x \vee y = sup\{x, y\}$, and $x \wedge y = inf\{x, y\}$.*

**Definition 2.5.** [6] Let $U$ be an initial universe set and $E$ be a parameters set. Let $\mathcal{F}(U)$ be the power set of all fuzzy subsets of $U$, $B \subset E$. Then a pair $(F, B)$ is called a fuzzy soft set over $U$, where $F : A \to \mathcal{F}(U)$ is a mapping.

**Claim 2.** *Let $(F, E)$ be a fuzzy soft set. 'x $\in U, x \in (F, E)$ iff $\forall e \in E, x \in F(e) \subseteq U$' will be described by the membership of 'x belongs to $(F, E)$' and the symbol $m(F(e))(x) = \{m(F(e))(x) \mid x \in U, e \in E\}$ represents its membership. For convenience, we can use table to show fuzzy soft set $(F, E)$.*

**Example 2.1.** Assume fuzzy soft set $(F, E)$ describes the attraction of the considerable smart phone with respect to the given parameters. $U = \{s_1, s_2, s_3, s_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ in which $e_1$:fashionable, $e_2$:powerful function, $e_3$: pixel height, $e_4$: large screen, $e_5$: running memory, $e_6$: battery capacity. The fuzzy soft set $(F, E)$ are shown in Table 1.

**Table 1. The fuzzy soft set $(F, B)$**

|       | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|-------|-------|-------|-------|-------|
| $e_1$ | 0.5   | 0.7   | 0.3   | 0.4   |
| $e_2$ | 0.7   | 0.8   | 0.9   | 0.2   |
| $e_3$ | 0.9   | 0.7   | 0.4   | 0.4   |
| $e_4$ | 0.7   | 0.8   | 0.2   | 0.1   |
| $e_5$ | 0.6   | 0.9   | 0.5   | 0.5   |
| $e_6$ | 1     | 1     | 0.7   | 0.9   |

**Table 2. The fuzzy soft set $(F_1, B_1)$ and $(F_2, B_2)$**

|       | $(F_1, B_1)$ | | | | $(F_2, B_2)$ | | | |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|       | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_1$ | $e_3$ | $e_5$ | $e_6$ |
| $h_1$ | 0.3   | 0.7   | 0.5   | 0.3   | 0.4   | 0.3   | 0.5   | 0.5   |
| $h_2$ | 0.4   | 0.7   | 0.7   | 0.3   | 0.5   | 0.4   | 0.7   | 0.7   |
| $h_3$ | 0.4   | 0.8   | 0.7   | 0.5   | 0.5   | 0.4   | 0.7   | 0.8   |
| $h_4$ | 0.5   | 0.7   | 0.7   | 0.4   | 0.7   | 0.5   | 0.7   | 0.7   |
| $h_5$ | 0.7   | 0.9   | 0.7   | 0.7   | 0.7   | 0.8   | 0.8   | 0.8   |
| $h_6$ | 0.8   | 0.8   | 0.8   | 0.7   | 0.9   | 0.5   | 0.7   | 0.8   |
| $h_7$ | 0.9   | 0.9   | 1     | 0.9   | 1     | 0.9   | 0.8   | 0.9   |

We express the fuzzy soft subset $F(e) \subseteq U, (e \in A)$ in a similar way to the fuzzy set . For example, $F(e_1) = \{\frac{s_1}{0.5}, \frac{s_2}{0.7}, \frac{x_3}{0.3}, \frac{s_4}{0.4}\} = \frac{s_1}{0.5} + \frac{s_2}{0.7} + \frac{s_3}{0.3} + \frac{s_4}{0.4} = (0.5, 0.7, 0.3, 0.4)$, which $m(F(e_1))(s_1) = 0.5$ means that the membership $s_1$ belongs to the fashionable appearance smart phone is 0.5.

**Definition 2.6.** [16] Let $(L, M, F)$ be a fuzzy soft formal context, in which $L$ is the universe, $\mathcal{F}(L)$ be the class of all fuzzy subset of $L$, and $(F, B)$ be a fuzzy soft set over $L$, that is, $\forall e \in B, F(e) \in \mathcal{F}(L)$.

(i) For all $\alpha \in [0, 1]$, define
$$(F, B)_\alpha = \{((F(e))_\alpha, e) \mid e \in B\},$$

where $(F(e))_\alpha = \{x \in L | m(F(e))(x) \geq \alpha\}$ is $\alpha$-level set, in other words, $(F(e))_\alpha$ is some objects set whose membership is bigger than a given $\alpha \in [0,1]$. Then $(F, B)_\alpha$ is called the $\alpha$-level fuzzy soft set, also called the $\alpha$-cut fuzzy soft set.

(ii) Define the preference order $\preceq$ on $L$ as: $\forall h, k \in L$, $h \preceq k$    if  and only if    $m(F(e))_h \preceq m(F(e))_k$, $\forall\ e \in B$. Clearly, $\preceq$ is a partial order, and stipulate: if $h \preceq k$, then $h$ is in the below of $k$ in the lattice structure.

(iii) If $\forall e \in B, \exists \alpha \in [0,1]$, such that $F(e) \subseteq L$, $(F(e))_\alpha$ is a sublattice of $L$, then $(F, B)$ is a $\alpha$-level fuzzy soft lattice over $L$.

(iv) If $\forall e \in B, \forall \alpha \in [0,1]$, such that $F(e) \subseteq L$, $(F(e))_\alpha$ is a sublattice of $L$, then $(F, B)$ is simply called a fuzzy soft lattice over $L$.

**Remark 1.** *Obviously, the $\alpha$-level fuzzy soft set is the classical soft set, and the membership of $x$ belongs to $\alpha$-level fuzzy soft set $(F, B)$ is as following:*

$$m(F, B)(x) = \left\{ \begin{array}{ll} 1, & if\ \ m(F(e))(x) \geq \alpha \\ 0, & if\ \ m(F(e))(x) < \alpha. \end{array} \right.$$

**Definition 2.7.** [6] Let $U$ be an initial universe set and $E$ be a parameters set. Let $\mathcal{F}(U)$ be the power set of all fuzzy subsets of $U$, $A, B \subset E$, $(F, A)$ and $(G, B)$ be two fuzzy soft sets over $U$ .

(i) If $A \subseteq B$, and $\forall e \in A$, having $F(e) \subseteq G(e)$, then $(F, A)$ is a fuzzy soft subset of $(G, B)$, denoted as $(F, A) \widetilde{\subset} (G, B)$.

(ii) $(F, A)$ and $(G, B)$ are said fuzzy soft equal, if $(F, A) \widetilde{\subset} (G, B)$, and $(G, B) \widetilde{\subset} (F, A)$. We simply denote by $(F, A) = (G, B)$.

(iii) The complement of $(F, A)$ is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \neg A)$, where $F^c : \neg A \rightarrow \mathcal{F}(U)$, and $F^c(e) = F(\neg e)$, $\forall e \in \neg A$. In here, we define $F^c(e) = F(\neg e) = 1 - F(e), \forall e \in A$.

For example, in the Example 2.1, $F^c(e_1) = \{\frac{s_1}{0.5}, \frac{s_2}{0.3}, \frac{s_3}{0.7}, \frac{e_4}{0.6}\} = \frac{s_1}{0.5} + \frac{s_2}{0.3} + \frac{s_3}{0.7} + \frac{s_4}{0.6} = (0.5, 0.3, 0.7, 0.6)$, which means the membership $e_1 \in \neg E$.

(iv) The fuzzy soft union  of $(F, A)$ and $(G, B)$ is the fuzzy soft set $(H, C)$, where $C = A \cup B$, and $\forall e \in C$, denoted as $(F, A) \widetilde{\cup} (G, B) = (H, C) = (H, A \cup B)$, where

$$H(e) = \left\{ \begin{array}{ll} F(e), & if\ \ e \in A - B \\ G(e), & if\ \ e \in B - A \\ F(e) \cup G(e), & if\ \ e \in A \cap B \end{array} \right.$$

(v) The fuzzy soft intersection of $(F, A)$ and $(G, B)$ is the fuzzy soft set $(H, C)$ is denoted as $(F, A) \sqcap (G, B)$ and is defined as $(F, A) \sqcap (G, B) = (H, C)$, where $C = A \cap B$, and $\forall e \in C, H(e) = F(e) \cap G(e)$.

# 3 The Operations for the $\alpha$-level Fuzzy Soft Sets (Lattices)

**Definition 3.1.** Let $(L, M, F)$ be a fuzzy soft formal context in which $L$ is the universe, $(F, B)$ is a fuzzy soft set over $L$. Let $\mathcal{F}(L)$ be the power set of all fuzzy subsets of $L$, $B_1, B_2 \subset M$, $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha (\alpha \in [0,1])$ be two $\alpha$-level fuzzy soft sets over $L$.

(i) If $B_1 \subseteq B_2$ , and $\forall e \in A$, having  $(F_1(e))_\alpha \subseteq (F_2(e))_\alpha$ (i.e.$\forall x \in L$, $m(F_1(e))_x \leq m(F_1(e))_x$), then $(F_1, B_1)_\alpha$ is a $\alpha$-level fuzzy soft subset  of $(F_2, B_2)_\alpha$, denoted as $(F_1, B_1)_\alpha \sqsubseteq (F_2, B_2)_\alpha$.

(ii) $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ are said $\alpha$-level fuzzy soft equal , if $(F_1, B_1)_\alpha \sqsubseteq (F_2, B_2)_\alpha$, and $(F_2, B_2)_\alpha \sqsubseteq (F_1, B_1)_\alpha$. We simply denote by $(F_1, B_1)_\alpha = (F_2, B_2)_\alpha$.

   **Denotation** If $m(F(e))(x) \geq \alpha, \forall e \in B \subseteq M$, then the object $x$ belongs to $(F, B)_\alpha$ (i.e. $x \in (F, B)_\alpha$).

   $(F(x))_\alpha = \{e \in B \subseteq M \mid x \in [F(e)]_\alpha\} = \{e \in B \subseteq M \mid m(F(e))(x) \geq \alpha\}$.

   **Equivalent Description** If there is a bijective mapping $\sigma : (F_1, B_1)_\alpha \to (F_2, B_2)_\alpha$, s.t.$\sigma(F_1(x))_\alpha = (F_2(x))_\alpha$, in which $(F_i(x))_\alpha = \{e \in B_i \mid x \in (F_i(e))_\alpha, \alpha \in [0,1]\}(i = 1, 2)$, then $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ are $\alpha$-level soft equal.

(iii) The $\alpha$-level fuzzy soft complement of $(F_1, B_1)_\alpha$ is denoted by $(F_1, B_1)_\alpha^c$ and defined by $(F_1, B_1)_\alpha^c = (F_1^c, B_1)_\alpha$, where $F_1^c : \neg B_1 \to \mathcal{F}(L)$, and $(F_1^c(e))_\alpha = (F_1(\neg e))_\alpha, \forall e \in \neg B_1$.

   For example, in Example 2.1, $(F^c(e_1))_{0.6} = \{s_3, s_4\}$.

(iv) The $\alpha$-level fuzzy soft union of $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ is the $\alpha$-level fuzzy soft set $(H, C)_\alpha$, where $C = B_1 \cup B_2$, and $\forall e \in C$, denoted as $(F_1, B_1)_\alpha \widetilde{\cup} (F_2, B_2)_\alpha = (H, C)_\alpha = \{((H(e))_\alpha, e) | e \in B_1 \cup B_2\}$, where

$$(H(e))_\alpha = \begin{cases} (F_1(e))_\alpha, & \text{if } e \in B_1 - B_2 \\ (F_2(e))_\alpha, & \text{if } e \in B_2 - B_1 \\ (F_1(e))_\alpha \cup (F_2(e))_\alpha, & \text{if } e \in B_1 \cap B_2 \end{cases}$$

(v) The $\alpha$-level fuzzy soft intersection of $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ is the $\alpha$-level fuzzy soft set $(H, C)_\alpha$ and denoted as $(F_1, B_1)_\alpha \sqcap (F_2, B_2)_\alpha = (H, C)_\alpha = \{((H(e))_\alpha, e) | e \in C = B_1 \cap B_2\}$, and $\forall e \in C, (H(e))_\alpha = (F_1(e))_\alpha \cap (F_2(e))_\alpha$.

**Remark 2.** *(i) As mentioned earlier, the difference between $\alpha$-level fuzzy soft lattices and $\alpha$-level fuzzy soft sets is that $F(e)_\alpha$ should be a sublattice of $L$, so their operation mode is the same. That is to say, if $(L, M, F)$ is a fuzzy soft formal context in which $L$ is a lattice, $\exists \alpha \in [0, 1]$, such that $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ are sublattices of $L$, then the above operations become the operations of $\alpha$-level fuzzy soft lattices.*

*(ii) In the above binary operations, the value of $\alpha$ can be the same or different, for example, $(F_1, B_1)_{0.8} \sqcap (F_2, B_2)_{0.8}$ or $(F_1, B_1)_{0.5} \widetilde{\cup} (F_2, B_2)_{0.8}$, and so on.*

*(iii) For $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$, they can all be $\alpha$-level fuzzy soft sets or all be $\alpha$-level fuzzy soft lattices, or one is $\alpha$-level fuzzy soft lattice and another is $\alpha$-level fuzzy soft set.*

**Example 3.1.** $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ which is the set of the attract considerable houses. $M = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ are houses attributes set in which $e_1$:cheap, $e_2$:modern, $e_3$:nearby the station, $e_4$:nearby the school, $e_5$:nearby the supermarkets, $e_6$:in the green surroundings. And $B_1 = \{e_1, e_2, e_3, e_4\} \subseteq M$ is the attributes subset that the young couples pay more attention to. $B_2 = \{e_1, e_3, e_5, e_6\} \subseteq M$ is the attributes subset that the older people follow most interest. The fuzzy soft sets $(F_1, B_1)$ and $(F_2, B_2)$ can be described using Table 2. Their $\alpha$-level fuzzy soft set $(F_1, B_1)_\alpha$ and $(F_1, B_1)_\alpha$ are as Table 3 and Table 4.

**Result 1.** *From Table 3 and Table 4, we can find $\exists \alpha = 0.7, (F_1, B_1)_{0.7}$ is not $\alpha$-level fuzzy soft lattice. Because $(F(e_1))_\alpha = \{h_5, h_6, h_7\}$, $h_5 \wedge h_6 = \{h_3, h_4\}$, but $h_3, h_4 \notin \{h_5, h_6, h_7\}$. And $\forall \alpha \in [0, 1], (F_2, B_2)_\alpha$ is $\alpha$-level fuzzy soft lattice, so $(F_2, B_2)_\alpha$ is a fuzzy soft lattice.*

$(F_1, B_1)_{0.8} \sqcap (F_2, B_2)_{0.8} = \{(\{h_6, h_7\}, e_1), (\{h_7\}, e_3)\}$,

$(F_1, B_1)_{0.8} \widetilde{\cup} (F_2, B_2)_{0.8} = \{(\{h_6, h_7\}, e_1), (\{h_3, h_5, h_6, h_7\}, e_2), (\{h_5, h_6, h_7\}, e_3),$
$(\{h_7\}, e_4), (\{h_5, h_7\}, e_5), (\{h_3, h_5, h_6, h_7\}, e_6)\}$,

$(F_1, B_1)_{0.5} \sqcap (F_2, B_2)_{0.7} = \{(\{h_4, h_5, h_6, h_7\}, e_1), (\{h_5, h_7\}, e_3)\}$,

$(F_1, B_1)_{0.8} \widetilde{\cup} (F_2, B_2)_{0.4} = \{(L, e_1), (\{h_3, h_5, h_6, h_7\}, e_2), (\{h_2, h_3, h_4, h_5, h_6, h_7\}, e_3), (\{h_7\}, e_4),$
$(L, e_5), (L, e_6)\}$.

**Table 3. The $\alpha$-level fuzzy soft set $(F_1, B_1)_\alpha$**

| | $(F_1, B_1)_\alpha$ | | | |
|---|---|---|---|---|
| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
| 0.9 | $\{h_7\}$ | $\{h_5, h_7\}$ | $\{h_7\}$ | $\{h_7\}$ |
| 0.8 | $\{h_6, h_7\}$ | $\{h_3, h_5, h_6, h_7\}$ | $\{h_6, h_7\}$ | $\{h_7\}$ |
| 0.7 | ${\color{red}\{h_5, h_6, h_7\}}$ | $L$ | $\{h_2, h_3, h_4, h_5, h_6, h_7\}$ | ${\color{red}\{h_5, h_6, h_7\}}$ |
| 0.5 | $\{h_4, h_5, h_6, h_7\}$ | $L$ | $L$ | $\{h_3, h_5, h_6, h_7\}$ |
| 0.4 | $\{h_2, h_3, h_4, h_5, h_6, h_7\}$ | $L$ | $L$ | $\{h_3, h_4, h_5, h_6, h_7\}$ |
| 0.3 | $L$ | $L$ | $L$ | $L$ |

**Table 4. The $\alpha$-level fuzzy soft set $(F_2, B_2)_\alpha$**

| | $(F_2, B_2)_\alpha$ | | | |
|---|---|---|---|---|
| | $e_1$ | $e_3$ | $e_5$ | $e_6$ |
| 0.9 | $\{h_6, h_7\}$ | $\{h_7\}$ | $\emptyset$ | $\{h_7\}$ |
| 0.8 | $\{h_6, h_7\}$ | $\{h_5, h_7\}$ | $\{h_5, h_7\}$ | $\{h_3, h_5, h_6, h_7\}$ |
| 0.7 | $\{h_4, h_5, h_6, h_7\}$ | $\{h_5, h_7\}$ | $\{h_2, h_3, h_4, h_5, h_6, h_7\}$ | $\{h_2, h_3, h_4, h_5, h_6, h_7\}$ |
| 0.5 | $\{h_2, h_3, h_4, h_5, h_6, h_7\}$ | $\{h_4, h_5, h_6, h_7\}$ | $L$ | $L$ |
| 0.4 | $L$ | $\{h_2, h_3, h_4, h_5, h_6, h_7\}$ | $L$ | $L$ |
| 0.3 | $L$ | $L$ | $L$ | $L$ |

From Table 5, we can get $(F_1, B_1)_{0.4}^c = \{(\{h_1, h_2, h_3, h_4\}, e_1), (\emptyset, e_2), (\{h_1\}, e_3), (\{h_1, h_2, h_3, h_4\}, e_4)\}$. If the membership is larger and the house is more popular, then according to their membership, we can define the preference order as $h_i \preceq h_j$ iff $m(F(e))(h_i) \leq m(F(e))(h_j), \forall e \in M$ , and their order structure forms as Fig. 1. and $(F_1, B_1)^c$ forms as Table 5.

**Table 5. The soft complement set of $(F_1, B_1)$**

| | $(F_1, B_1)^c$ | | | |
|---|---|---|---|---|
| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
| $h_1$ | 0.7 | 0.3 | 0.5 | 0.7 |
| $h_2$ | 0.6 | 0.3 | 0.3 | 0.7 |
| $h_3$ | 0.6 | 0.2 | 0.3 | 0.5 |
| $h_4$ | 0.5 | 0.3 | 0.3 | 0.6 |
| $h_5$ | 0.3 | 0.1 | 0.3 | 0.3 |
| $h_6$ | 0.2 | 0.2 | 0.2 | 0.3 |
| $h_7$ | 0.1 | 0.1 | 0 | 0.1 |

**Table 6. The fuzzy soft sets $(F_1, B_1)$ and $(F_2, B_2)$**

| | $B_1$ | | | $B_2$ | |
|---|---|---|---|---|---|
| | $e_1$ | $e_2$ | $e_3$ | $e_2$ | $e_3$ |
| $a$ | 0.2 | 0.2 | 0.3 | 0.3 | 0.3 |
| $b$ | 0.2 | 0.5 | 0.3 | 0.3 | 0.7 |
| $c$ | 0.5 | 0.2 | 0.3 | 0.5 | 0.3 |
| $d$ | 0.5 | 0.7 | 0.3 | 0.9 | 0.7 |
| $e$ | 0.7 | 0.5 | 0.3 | 0.5 | 0.9 |
| $f$ | 0.9 | 0.9 | 1 | 0.9 | 0.9 |

**Result 2.** *From Fig. 1, we can know the house $h_7$ will be given priority to both the young couples and the older people.*

**Remark 3.** *The soft complement $\alpha-$level fuzzy soft lattice is not closed, that is, $\exists \alpha \in [0,1], (F_1, B_1)_\alpha$ is $\alpha-$level fuzzy soft lattice and $(F_1, B_1)_\alpha^c$ is not.*

*We can find that $(F_1, B_1)_{0.4}$ is $\alpha$-level fuzzy soft lattice, but $(F_1, B_1)_{0.4}^c = \{(\{h_1, h_2, h_3, h_4\}, \neg e_1), (\emptyset, \neg e_2), (\{h_1\}, \neg e_3), (\{h_1, h_2, h_3, h_4\}, \neg e_4)\}$ from Table 5, $\{h_1, h_2, h_3, h_4\}$ is not lattice, because $h_3 \vee h_4 = \{h_5, h_6\}$, but $h_5, h_6 \notin \{h_1, h_2, h_3, h_4\}$.*

**Proposition 3.1.** *Let $(F, B)_\alpha$ be a $\alpha$-level fuzzy soft set over $(L, M, F)$. If $\alpha_1 \leq \alpha_2 \in [0,1]$, then $(F, B)_{\alpha_2} \sqsubseteq (F, B)_{\alpha_1}$.*

*Proof.*
$$m(F(e))_x \geq \alpha_2 \geq \alpha_1, x \in L \ \forall e \in B$$
$$\implies \quad x \in (F, B)_{\alpha_2}, \text{ having } x \in (F, B)_{\alpha_1}$$
$$\implies \quad (F(e))_{\alpha_2} \subseteq (F(e))_{\alpha_1}, \ \forall e \in B, \ i.e. \ (F, B)_{\alpha_2} \sqsubseteq (F, B)_{\alpha_1}.$$

$\square$

**Remark 4.** *(i) The operation result of $\alpha$-level fuzzy soft union (intersection) must be the $\alpha$-level fuzzy soft sets, not $\alpha$-level fuzzy soft lattice.*

*(ii) Even if $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ are $\alpha$-level fuzzy soft lattices over $(L, M, F)$, $(F_1, B_1)_\alpha \sqcap (F_2, B_2)_\alpha$ and $(F_1, B_1)_\alpha \widetilde{\cup} (F_2, B_2)_\alpha$ can not be $\alpha$-level fuzzy soft lattice.*

**Example 3.2.** Let the universe $L = \{a, b, c, d, e, f\}$ be a lattice (its figure as Fig.2). $E = \{e_1, e_2, e_3\}$ be attributes set. Suppose that $B_1 = \{e_1, e_2, e_3\} \subseteq E$, $B_2 = \{e_2, e_3\} \subseteq E$, $(F_1, B_1), (F_2, B_2)$ are fuzzy soft sets as the Table 6.

Given the different $\alpha$, the corresponding $\alpha$-level sets of all fuzzy soft sets as Table 7.

**Table 7. The $\alpha$-level fuzzy soft sets $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$**

| $\alpha$ | $(F_1, B_1)_\alpha$ | | | $(F_2, B_2)_\alpha$ | |
|---|---|---|---|---|---|
| | $(F_1(e_1))_\alpha$ | $(F_1(e_2))_\alpha$ | $(F_1(e_3))_\alpha$ | $(F_2(e_2))_\alpha$ | $(F_2(e_3))_\alpha$ |
| 0.9 | $\{f\}$ | $\{f\}$ | $\{f\}$ | $\{d, f\}$ | $\{e, f\}$ |
| 0.7 | $\{e, f\}$ | $\{d, f\}$ | $\{f\}$ | $\{d, f\}$ | $\{b, d, e, f\}$ |
| 0.5 | $\{c, d, e, f\}$ | $\{b, d, e, f\}$ | $\{f\}$ | $\{c, d, e, f\}$ | $\{b, d, e, f\}$ |
| 0.3 | $\{b, d, e, f\}$ | $\{a, d, e, f\}$ | $L$ | $L$ | $L$ |
| 0.2 | $L$ | $L$ | $L$ | $L$ | $L$ |

From the above table, we can find that for all given $\alpha$, $(F_1(e_i))_\alpha$ and $(F_2(e_i))_\alpha (i = 1, 2, 3)$ are all sublattice of $L$ (their lattice structure as Fig.2(1)), that is, $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ are $\alpha$-level fuzzy soft lattices, however, $(F_1, B_1)_\alpha \sqcap (F_2, B_2)_\alpha = \{(\{d, e, f\}, e_2), (\{f\}, e_3)\}$ is not $\alpha$-level fuzzy soft lattice because of $d \wedge e = \{b, c\}$, but $b, c \notin \{d, e, f\}$(i.e. $\{d, e, f\}$ is not the sublattice of $L$)(see Fig. 2(2)). $(F_1, B_1)_\alpha \widetilde{\cup} (F_2, B_2)_\alpha = \{(\{c, d, e, f\}, e_1), (\{b, c, d, e, f\}, e_2), (L, e_3)\}$ is not $\alpha$-level fuzzy soft lattice because of $b \wedge c = \{a\}$, but $a \notin \{b, c, d, e, f\}$ (i.e.$\{b, c, d, e, f\}$ is not the sublattice of $L$)(see Fig. 2(3)).

**Remark 5.** *If the universe $L$ is a complete lattice, $\alpha \in [0,1]$, $(F_1, B_1)_\alpha$ and $(F_2, B_2)_\alpha$ are $\alpha$-level fuzzy soft lattices, then $(F_1, B_1)_\alpha \sqcap (F_2, B_2)_\alpha$ and $(F_1, B_1)_\alpha \widetilde{\cup} (F_2, B_2)_\alpha$ are also $\alpha$-level fuzzy soft lattice. That is, the completeness of $L$ ensures that its sublattices are still $\alpha$-level fuzzy soft lattices after soft intersection and soft union.*

**Proposition 3.2.** *Let $(F_1, B_1)$ and $(F_2, B_2)$ be fuzzy soft sets over $(L, M, F)$, and $(F_1, B_1) \widetilde{\subset} (F_2, B_2)$, then $\forall \alpha \in [0,1]$, $(F_1, B_1)_\alpha \sqsubseteq (F_2, B_2)_\alpha$.*

*Proof.*

$$\begin{aligned} & (F_1, B_1)\widetilde{\subset}(F_2, B_2) \\ \Longrightarrow \quad & F_1(e) \subseteq F_2(e) \subseteq L, \ \forall e \in B_1 \subseteq B_2 \subseteq M \\ \Longrightarrow \quad & \forall \alpha \in [0,1], \text{if } m((F_1(e))(x) \geq \alpha, \text{then } m((F_2(e))(x) \geq \alpha, \forall x \in L, \forall e \in B_1 \subseteq B_2 \subseteq M \\ \Longrightarrow \quad & \{(F_1(e))_\alpha | e \in B_1 \subseteq B_2 \subseteq M\} \subseteq \{(F_2(e))_\alpha | e \in B_1 \subseteq B_2 \subseteq M\} \\ \Longrightarrow \quad & (F_1, B_1)_\alpha \sqsubseteq (F_2, B_2)_\alpha. \end{aligned}$$

$\square$

**Proposition 3.3.** *If $(F_1, B_1)_\alpha$ is $\alpha$-level fuzzy soft subset of $(F_2, B_2)_\alpha$ then $(F_1, B_1)_\alpha \sqcap (F_2, B_2)_\alpha = (F_1, B_1)_\alpha$ and $(F_1, B_1)_\alpha \widetilde{\cup} (F_2, B_2)_\alpha = (F_2, B_2)_\alpha$.*

*Proof.*

$$\begin{aligned} & (F_1, B_1)_\alpha \sqsubseteq (F_2, B_2)_\alpha \\ \Longrightarrow \quad & m((F_1(e))_x \leq m((F_2(e))_x, \forall x \in L, \forall e \in B_1 \subseteq B_2 \\ \Longrightarrow \quad & \{(F_1(e))_\alpha | e \in B_1\} \cap \{(F_2(e))_\alpha | e \in B_1\} = \{(F_1(e))_\alpha | e \in B_1\}, \forall e \in B_1 \cap B_2 = B_1, \text{ and} \\ & \{(F_1(e))_\alpha | e \in B_2\} \cup \{(F_2(e))_\alpha | e \in B_2\} = \{(F_2(e))_\alpha | e \in B_2\}, \forall e \in B_1 \cup B_2 = B_2 \\ \Longrightarrow \quad & (F_1, B_1)_\alpha \sqcap (F_2, B_2)_\alpha = (F_1, B_1)_\alpha, (F_1, B_1)_\alpha \widetilde{\cup} (F_2, B_2)_\alpha = (F_2, B_2)_\alpha. \end{aligned}$$

$\square$

**Proposition 3.4.** *If $(F, B)_{\alpha_1}$ and $(F, B)_{\alpha_2}$ are $\alpha$-level fuzzy soft lattices, then $(F, B)_{\alpha_1} \sqcap (F, B)_{\alpha_2}$ and $(F, B)_{\alpha_1} \widetilde{\cup} (F, B)_{\alpha_2}$ are also $\alpha$-level fuzzy soft lattice.*

*Proof.* $\forall \alpha_1, \alpha_2 \in [0,1]$, if $\alpha_1 = \alpha_2$, by Proposition 2, the result holds.

Without less generalization, let

$$\begin{aligned} \alpha_1 > \alpha_2, \text{using proposition 2} \quad \Longrightarrow \quad & (F, B)_{\alpha_1} \sqsubseteq (F, B)_{\alpha_2} \\ \Longrightarrow \quad & (F, B)_{\alpha_1} \sqcap (F, B)_{\alpha_2} = (F, B)_{\alpha_1}, (F, B)_{\alpha_1} \widetilde{\cup} (F, B)_{\alpha_2} = (F, B)_{\alpha_2}. \end{aligned}$$

Hence, $(F, B)_{\alpha_1} \sqcap (F, B)_{\alpha_2}$ and $(F, B)_{\alpha_1} \widetilde{\cup} (F, B)_{\alpha_2}$ are also $\alpha$-level fuzzy soft lattice. $\square$

# 4 Decomposition Theorem for Fuzzy Soft Sets

**Definition 4.1.** Let $(L, M, F)$ be a fuzzy soft formal context, $(F, B)$ be a fuzzy soft set. That is, $\forall e \in B, F(e) \in \mathcal{F}(L)$, $\alpha \in [0,1]$, denote $\alpha(F, B) = \{(\{\alpha \wedge m(F(e)(x)\}, e) | e \in B, x \in L\}$, and $\alpha(F, B)$ is called the multiplication of $\alpha$ and fuzzy soft set$(F, B)$.

**Remark 6.** *Clearly, $\alpha(F, B)$ is a fuzzy soft set, and $\forall x \in L$, the membership of $x$ belonging to $\alpha(F, B)$ is :*

$$m(\alpha F(e)_\alpha(x)) = \begin{cases} \alpha, & \text{if } x \in (F, B)_\alpha \\ 0, & \text{if } x \notin (F, B)_\alpha \end{cases}$$

**Theorem 4.1.** *Let $(F, B)$ be s fuzzy soft set over $(L, M, F)$, then $(F, B) = \widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, B)_\alpha$.*

*Proof.*

$$\begin{aligned} (\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, B)_\alpha)(x) &= \bigvee_{\alpha \in [0,1]} (\alpha \wedge m(F(e))_\alpha(x)), \quad e \in B \\ &= max\left(\bigvee_{m(F(e))_\alpha(x) \geq \alpha} (\alpha \wedge m(F(e))_\alpha(x)), \bigvee_{m(F(e))_\alpha(x) < \alpha} (\alpha \wedge m(F(e))_\alpha(x))\right), \quad e \in B \\ &= max\left(\bigvee_{m(F(e))_\alpha(x) \geq \alpha} (\alpha \wedge 1), \bigvee_{m(F(e))_\alpha(x) < \alpha} (\alpha \wedge 0)\right), \quad e \in B \\ &= max\left(\bigvee_{m(F(e))_\alpha(x) \geq \alpha} (\alpha), \bigvee_{m(F(e))_\alpha(x) < \alpha} (0)\right), \quad e \in B \\ &= max\left(m(F(e))(x), 0\right), \quad e \in B \\ &= m(F(e))(x), \quad e \in B \\ &= (F, B)(x) \end{aligned}$$

$\square$

**Corollary 4.1.** *Let* $(F, B)$ *be s fuzzy soft set over* $(L, M, F)$, *then* $(F, B) = \widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, B)$.

*Proof.* It is clear from the definition for $\alpha(F, B)$ and the above theorem. □

**Example 4.1.** Let $(L, M, F)$ be a fuzzy soft context, in which $L = \{a, b, c\}, M = \{e_1, e_2, e_3\}$, fuzzy soft set $(F, M)$ and $\alpha(F, M)$ are as the following Table 8, $\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, M)$ are as Table 9.

**Table 8. The fuzzy soft set** $(F, M)$ **and** $\alpha(F, M)$

|   | $(F, M)$ | | | $0.9(F, M)$ | | | $0.7(F, M)$ | | | $0.6(F, M)$ | | | $0.5(F, M)$ | | |
|---|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|   | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ | $e_1$ | $e_2$ | $e_3$ |
| a | 0.5 | 0.7 | 0.9 | 0.5 | 0.7 | 0.9 | 0.5 | 0.7 | 0.7 | 0.5 | 0.6 | 0.6 | 0.5 | 0.5 | 0.5 |
| b | 0.5 | 0.9 | 0.9 | 0.5 | 0.9 | 0.9 | 0.5 | 0.7 | 0.7 | 0.5 | 0.6 | 0.6 | 0.5 | 0.5 | 0.5 |
| c | 0.6 | 0.7 | 0.9 | 0.6 | 0.7 | 0.9 | 0.6 | 0.7 | 0.7 | 0.6 | 0.6 | 0.6 | 0.5 | 0.5 | 0.5 |

**Table 9. The fuzzy soft set** $\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, M)$

|   | $\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, M)$ | | |
|---|------|------|------|
|   | $e_1$ | $e_2$ | $e_3$ |
| $a$ | 0.5 | 0.7 | 0.9 |
| $b$ | 0.5 | 0.9 | 0.9 |
| $c$ | 0.6 | 0.7 | 0.9 |

**Table 10. The fuzzy soft set** $(F, B)$

|   | $(F, B)$ | | | |
|---|------|------|------|------|
|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
| $a$ | 0.5 | 0.9 | 0.3 | 0.2 |
| $b$ | 0.3 | 0.2 | 0.2 | 0.1 |
| $c$ | 0.7 | 0.7 | 0.3 | 0.2 |
| $d$ | 0.9 | 1.0 | 0.5 | 0.3 |

Hence, $\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, M) = (F, M)$.

# 5  A Simple Application in Skin Disease Diagnosis

**Example 5.1.** Consider the patients have got the symptom of dermatitis.

Let $(L, M, F)$ be a fuzzy soft context, $M = \{e_1, e_2, e_3, e_4\}$ is the set of patients, $L = \{a, b, c, d\}$ is the universe, where $a$ indicates that skin has red spots, $b$ indicates that skin has parts of itch, $c$ shows that the surface of the skin has edema, $d$ shows that skin has parts of blisters.

As we know, when judging whether skin has dermatitis or not, we should give priority to symptom $d$, next $a$ and $c$, last is $b$. The Fig.3(1) is their order structure. Suppose that $B = M$, and fuzzy soft set $(F, B)$ (see the Table 10) describes the possibility that someone has got the symptom of dermatitis.

By decomposition theorem, we know $(F, B) = \widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, B)$, in fact , the table 10 and the table 12 also show this is a right result.

**Table 11.  The $\alpha$-level fuzzy soft set $(F, B)_\alpha$**

| $\alpha$ | $(F(e_1))_\alpha$ | $(F(e_2))_\alpha$ | $(F(e_3))_\alpha$ | $(F(e_4))_\alpha$ |
|---|---|---|---|---|
| | | $(F, B)_\alpha$ | | |
| 1 | $\emptyset$ | $\{d\}$ | $\emptyset$ | $\emptyset$ |
| 0.9 | $\{d\}$ | $\{a, d\}$ | $\emptyset$ | $\emptyset$ |
| 0.7 | $\{c, d\}$ | $\{a, c, d\}$ | $\{d\}$ | $\emptyset$ |
| 0.5 | $\{a, c, d\}$ | $\{a, c, d\}$ | $\{d\}$ | $\emptyset$ |
| 0.3 | $L$ | $\{a, c, d\}$ | $\{a, c, d\}$ | $\{d\}$ |
| 0.2 | $L$ | $L$ | $L$ | $\{a, c, d\}$ |
| 0.1 | $L$ | $L$ | $L$ | $L$ |

**Table 12.  The fuzzy soft set $\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, B)$**

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| | | $\widetilde{\bigcup}_{\alpha \in [0,1]} \alpha(F, B)$ | | |
| $a$ | 0.5 | 0.9 | 0.3 | 0.2 |
| $b$ | 0.3 | 0.2 | 0.2 | 0.1 |
| $c$ | 0.7 | 0.7 | 0.3 | 0.2 |
| $d$ | 0.9 | 1.0 | 0.5 | 0.3 |

**Result 3.**  *From the Table 11, we can find that there is $\alpha$-level fuzzy soft set $(F, B)_\alpha$ which is not $\alpha$-level fuzzy soft lattice, e.g. $\alpha = 0.5, F(e_1)_\alpha = \{a, c, d\}$ is not a lattice because $a \wedge c = \{b\}$, but $b \notin \{a, c, d\}$.*

**Result 4.**  *Form the lattice structure (see Fig. 3(2)), we can get the patient $e_4$ does not have the dermatitis, and the patient $e_2$ has the dermatitis.*
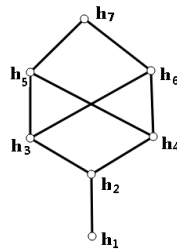


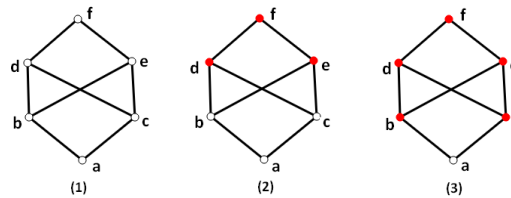**Fig. 1.  The order structure of Example 3.1**
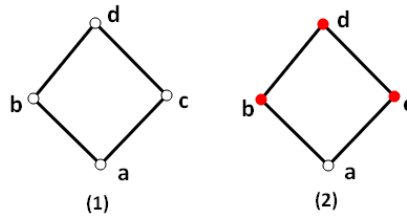


**Fig. 2.  The order structure of Example 3.2**

**Fig. 3. The order structure of Example 5.1**

# 6 Conclusion

From the above discussion, this paper mainly discusses the properties of $\alpha$-level fuzzy soft sets and $\alpha$-level fuzzy soft lattices. Firstly, we define the operations for $\alpha$-level fuzzy soft sets (lattices). Additional, it not only studies their properties, but also discusses their relationship and tests some properties not valid in the case discussed, but tenable in the classical soft set theory. Next, the decomposition theorem of the $\alpha$-level fuzzy soft sets is studied. Lastly, we give a simple application to help us to understand the $\alpha$-level fuzzy soft sets, and its decomposition theorem. Therefore, our research may provide a new idea for hierarchical ordering of the data and related academic field.

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# Competing Interests

Authors have declared that no competing interests exist.

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