academicJournals

Vol. 10(5), pp. 32-37, November 2017
DOI: 10.5897/AJMCSR2015.0625
Article Number: 7B9E0E966705
ISSN 2006-9731

http://www.academicjournals.org/AJMCSR

African Journal of Mathematics and Computer Science Research

Review

Adams completion and symmetric algebra

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Received 2 August, 2015; Accepted 22 September, 2015

Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context. In this paper, the symmetric algebra of a given algebra is shown to be the Adams completion of the algebra by considering a suitable set of morphisms in a suitable category.

Key words: Category of fraction, calculus of left fraction, symmetric algebra, tensor algebra, Adams completion.

INTRODUCTION

The notion of generalized completion (Adams completion) arose from a categorical completion process suggested by Adams (1973, 1975). Originally this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu et al. (1974) where an arbitrary category and an arbitrary set of morphisms of the category are considered; moreover they have also suggested the dual notion, namely the completion (Adams completion) of an object in a category.

The notion of Let $\mathcal{C}$ be an arbitrary category and S a set of morphisms of $\mathcal{C}$. Let $\mathcal{C}[S^{-1}]$ denote the category of fractions of $\mathcal{C}$ with respect S and $F\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the canonical function. Let $\mathcal{S}$ denote the category of sets and functions. Then for a given object Y of $\mathcal{C}$,

$$\mathcal{C}[S^{-1}](Y,-) : \mathcal{C} \to \mathcal{S}$$

defines a covariant function. If this function is representable by an object Y_S of $\mathcal{C}$, that is, $\mathcal{C}[S^{-1}](Y,-) \cong \mathcal{C}(Y_S,-)$. then Y_S is called the (generalized) Adams completion of Y with respect to the set of morphisms S or simply the S-*cocompletion* of Y. We shall often refer to Y_S as the completion of Y (Deleanu et al., 1974). Given a set S of morphisms of $\mathcal{C}$, the *saturation* $\bar{S}$ of S is defined as the set of all morphisms u in $\mathcal{C}$ such that $F(u)$ is an isomorphism in $\mathcal{C}[S^{-1}]$. S is said to be *saturated* if $S = \bar{S}$ (Deleanu et al., 1974).

Theorem 1

Behera and Nanda (1987) Let $\mathcal{C}$ be a complete small $\mathcal{U}$-category ($\mathcal{U}$ is a fixed Grothendeick universe) and S a set of morphisms of $\mathcal{C}$ that admits a calculus of left

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AMS Subject Classification: 15A72, 55P60.

fractions. Suppose that the following compatibility condition with co-product is satisfied. If each $s_i : X_i \to Y_i$, $i \in I$, is an element is of $\mathcal{U}$, then

$$\coprod_{i\in I} s_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$$

is an element of S. Then every object X of $\mathcal{C}$ has an Adams completion X_S with respect to the set of morphisms S.

Theorem 2

Let S be a set of morphisms of $\mathcal{C}$ admitting a calculus of left fractions. Then an object Y_S of $\mathcal{C}$ is the S-completion of the object Y with respect to S if and only if there exists a morphism $e : Y \to Y_S$ in $\bar{S}$ which is co-universal with respect to morphisms of S : given a morphism $s : Y \to Z$ in S there exists a unique morphism $t : Z \to Y_S$ in $\bar{S}$ such that $ts = e$. In other words, the following diagram is commutative (Deleanu et al., 1974):

$Y \overset{e}{\to} Y_S$

$s \downarrow \quad \nearrow t$

Z

Theorem 3

Let S be a set of morphisms in a category $\mathcal{C}$ admitting a calculus of left fractions. Let $e : Y \to Y_S$ be the canonical morphism as defined in Theorem 2, where Y_S is the S-completion of Y. Furthermore, let S_1 and S_2 be sets of morphisms in the category $\mathcal{C}$ which have the following properties (Behera and Nanda, 1987b):

S_1 and S_2 are closed under composition,
$fg \in S_1$ implies that $g \in S_1$,
$fg \in S_2$ implies that $f \in S_2$,
$S = S_1 \cap S_2$.
Then $e \in S$.

Symmetric algebra

Let K be a commutative ring. Let M be a K-module and $T(M)$ be the *tensor algebra* of M over K. $T(M)$ is a graded K-algebra with the graded piece of degree $n \geq 0$ being the additive subgroup $M^{\otimes n}$, which we denote by $T^n(M)$. The map $M^{\otimes n} \to T^n(M)$ defined by $m_1 \otimes \cdots \otimes m_n \to (0, \cdots \otimes m_1 \otimes \cdots \otimes m_n, 0, \cdots)$ is a morphism of K-modules, which gives an isomorphism of K-modules of M with its image $T^n(M)$ (Murfet, 2006). Let $A^n(M)$ denote the n-th symmetric algebra (Grinberg, 2013). The map $\rho_n : M^{\otimes n} \to A^n(M)$ is a subjective K-module homomorphism. We prove the following for our need.

Theorem 4

Let K be a commutative ring with unit 1. Let $f : M^{\otimes n} \to N^{\otimes n}$ be K-module isomorphism. Then f has the following property: given a module isomorphism $g : M^{\otimes n} \to T^n(M)$, there exists a unique module isomorphism $\theta : N^{\otimes n} \to T^n(M)$ such that $g = \theta f$, that is, the following diagram is commutative :

$M^{\otimes n} \overset{f}{\to} N^{\otimes n}$

$g \downarrow \quad \swarrow \theta$

$T^n(M)$

Proof 1

For $n_1 \otimes n_2 \otimes \cdots \otimes n_n \in N^{\otimes n}$, define
$\theta : N^{\otimes n} \to T^n(M)$
by the rule
$\theta(n_1 \otimes n_2 \otimes \cdots \otimes n_n) = gf^{-1}(n_1 \otimes n_2 \otimes \cdots \otimes n_n)$.

Clearly θ is well-defined, homomorphism, one-one and onto. We have

$(m_1 \otimes m_2 \otimes \cdots \otimes m_n)$
$= \theta\big(f(m_1 \otimes m_2 \otimes \cdots \otimes m_n)\big)$
$= gf^{-1}(f(m_1 \otimes m_2 \otimes \cdots \otimes m_n))$
$= g(m_1 \otimes m_2 \otimes \cdots \otimes m_n)$,

showing $g = \theta f$.

For showing the uniqueness of θ let there exist another $\theta' : N^{\otimes n} \longrightarrow T^n(M)$ such that $g = \theta' f$. Then

$$\begin{aligned} &\theta(n_1 \otimes n_2 \otimes \cdots \otimes n_n) \\ &= gf^{-1}(n_1 \otimes n_2 \otimes \cdots \otimes n_n) \\ &= \theta' f f^{-1}(n_1 \otimes n_2 \otimes \cdots \otimes n_n) \\ &= \theta'(n_1 \otimes n_2 \otimes \cdots \otimes n_n). \end{aligned}$$

This completes the proof.

Theorem 5

Let K be a commutative ring with unit 1. Let $M^{\otimes n}$ and $N^{\otimes n}$ be free K-modules and let $f : M^{\otimes n} \to N^{\otimes n}$ be a free K-module subjective homomorphism. Then f has the following property: given a free K-module subjective homomorphism $\rho_M : M^{\otimes n} \to A^n(M)$, there exists a unique free K-module subjective homomorphism $\varphi : N^{\otimes n} \to A^n(M)$ such that $\rho_M = \varphi f$, that is, the following diagram is commutative:

$$\begin{array}{ccc} M^{\otimes n} & \xrightarrow{f} & N^{\otimes n} \\ \rho_M \downarrow & \swarrow \varphi & \\ A^n(M) & & \end{array}$$

Proof 2

Theorem 4, there exists a unique K-module isomorphism $\psi : N^{\otimes n} \to T^n(M)$ such that $\psi f = g$:

$$\begin{array}{ccc} M^{\otimes n} & \xrightarrow{f} & N^{\otimes n} \\ g \downarrow & \swarrow \psi & \\ T^n(M) & & \end{array}$$

Where $g : M^{\otimes n} \to T^n(M)$ is a K-module isomorphism. Consider the diagram

$$\begin{array}{ccc} M^{\otimes n} & \xrightarrow{f} & N^{\otimes n} \\ g \downarrow & \swarrow \psi & \downarrow \varphi \\ T^n(M) & \xrightarrow[h]{} & A^n(M) \end{array}$$

Let $h = \rho_M g^{-1}$. For each $n \in N^{\otimes n}$, define $\varphi : N^{\otimes n} \to A^n(M)$ by the rule $\varphi(n) = h\psi(n)$. Then for each $m \in M^{\otimes n}$ we have

$$\varphi f(m) = h\psi(f(m)) = \rho_M g^{-1}\psi(f(m)) = \rho_M f^{-1}\psi^{-1}\psi(f(m)) = \rho_M f^{-1}(f(m)) = \rho_M(m)$$

Showing $\varphi f = \rho_M$. Clearly φ is subjective. For the uniqueness suppose there exists another $\varphi' : N^{\otimes n} \to A^n(M)$ such that $\rho_M = \varphi' f$. For any each $n \in N^{\otimes n}$ let $n = f(m)$, $m \in M^{\otimes n}$; thus $\varphi(n) = \varphi f(m) = \rho_M(m) = \varphi' f(m) = \varphi'(n)$, showing $\varphi = \varphi'$.This completes the proof.

The category $\mathcal{M}$

Let $\mathcal{U}$ be a fixed Grothendieck universe (Schubert, 1972). Let $\mathcal{M}$ denote the category of all free K-modules and free module homomorphisms where K is a commutative ring with unit 1. We assume that the underlying sets of the elements of $\mathcal{M}$ are elements of $\mathcal{U}$. Let S_n denote the set of all free K-module homomorphisms $f : M^{\otimes n} \longrightarrow N^{\otimes n}$ such that f is subjective.

Proposition

Let $\{s_i : X_i^{\otimes n} \to Y_i^{\otimes n},\ i \in I\}$ be a subset of S_n; where the index set I is an element of $\mathcal{U}$, then

$$\bigvee_{i \in I} s_i : \bigvee_{i \in I} X_i^{\otimes n} \to \bigvee_{i \in I} Y_i^{\otimes n}$$

is an element of S_n.

Proof 3

The proof is trivial.

We will show that the set S_n of free K-module homorphisms of the category $\mathcal{M}$ of free K-modules and free K-modules homomorphisms admits a calculus of left fraction.

Proposition

S_n admits a calculus of left fractions.

Proof 4

Since S_n consists of all subjective K-module homomorphisms in $\mathcal{M}$; clearly S_n is a closed family of morphisms of the category $\mathcal{M}$. We shall verify conditions (i) and (ii) of Theorem 1.3 ([6], p. 67). Let $M^{\otimes n}$, $N^{\otimes n}$ and $P^{\otimes n}$ be in $\mathcal{M}$. Let $u : M^{\otimes n} \longrightarrow N^{\otimes n}$ and $v : N^{\otimes n} \longrightarrow P^{\otimes n}$ be two free K-module homomorphisms of the category $\mathcal{M}$. We show that if $vu \in S_n$ and $u \in S_n$ then $v \in S_n$. Since $vu \in S_n$ and $u \in S_n$ we have $vu(M^{\otimes n}) = P^{\otimes n}$ and $u(M^{\otimes n}) = N^{\otimes n}$. Then $v(N^{\otimes n}) = v\left(u(M^{\otimes n})\right) = P^{\otimes n}$. So v is surjective. Hence condition (i) of Theorem 2 (Deleanu et al., 1974) holds. In order to prove condition (ii) of Theorem 2 (Deleanu et al., 1974) consider the diagram

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{f} & B^{\otimes n} \\ s\downarrow & & \\ C^{\otimes n} & & \end{array}$$

in $\mathcal{M}$ with $s \in S_n$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{f} & B^{\otimes n} \\ s\downarrow & & \downarrow t \\ C^{\otimes n} & \xrightarrow[g]{} & D^{\otimes n} \end{array}$$

in $\mathcal{M}$ with $t \in S_n$. Let
$D^{\otimes n} = (B^{\otimes n} \oplus C^{\otimes n})/N^{\otimes n}$

Where $N^{\otimes n}$ is a sub-module of $B^{\otimes n} \oplus C^{\otimes n}$ generated by

$\{(f(a_1 \otimes a_2 \otimes \cdots \otimes a_n), -s(a_1 \otimes a_2 \otimes \cdots \otimes a_n)) : a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}\}$.

Define $t : B^{\otimes n} \longrightarrow D^{\otimes n}$ by the rule
$t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = (b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) + N$
and $g : C^{\otimes n} \longrightarrow D^{\otimes n}$ by the rule
$g(c_1 \otimes c_2 \otimes \cdots \otimes c_n) = (0, c_1 \otimes c_2 \otimes \cdots \otimes c_n) + N$
.

Clearly, the two maps are well defined and homomorphisms. For any
$a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$, we have

$tf(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$
$= (f(a_1 \otimes a_2 \otimes \cdots \otimes a_n), 0)$
$= (0, s(c_1 \otimes c_2 \otimes \cdots \otimes c_n))$
$= gs(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$;

Thus $tf = gs$. Hence the diagram is commutative. In order to show t is subjective, take an element $d_1 \otimes d_2 \otimes \cdots \otimes d_n + N \in D^{\otimes n}$,

where
$d_1 \otimes d_2 \otimes \cdots \otimes d_n = (b_1 \otimes b_2 \otimes \cdots \otimes b_n,\ c_1 \otimes c_2 \otimes \cdots \otimes c_n)$
. Then

$d_1 \otimes d_2 \otimes \cdots \otimes d_n + N$
$= (b_1 \otimes b_2 \otimes \cdots \otimes b_n, c_1 \otimes c_2 \otimes \cdots \otimes c_n + N$
$=$
$(b_1 \otimes b_2 \otimes \cdots \otimes b_n, 0) + (0, c_1 \otimes c_2 \otimes \cdots \otimes c_n) + N$

$= t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + g(c_1 \otimes c_2 \otimes \cdots \otimes c_n)$
$=$
$t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + g(s(a_1 \otimes a_2 \otimes \cdots \otimes a_n))$
$=$
$t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + tf(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$
$=$
$t((b_1 \otimes b_2 \otimes \cdots \otimes b_n) + f(a_1 \otimes a_2 \otimes \cdots \otimes a_n))$

Thus t is an epimorphism. So $t \in S_n$.
Next let $u : B^{\otimes n} \to X^{\otimes n}$ and $v : C^{\otimes n} \to X^{\otimes n}$ be in category $\mathcal{M}$ such that $uf = vs$. Consider the following diagram

$$\begin{array}{cccc} A^{\otimes n} & \xrightarrow{f} & B^{\otimes n} & \\ s\downarrow & & \downarrow t & \\ & & & \searrow u \\ C^{\otimes n} & \xrightarrow[g]{} & D^{\otimes n} & \\ & & \theta \searrow & \\ & v \searrow & & \\ & & & X^{\otimes n} \end{array}$$

Define $\theta : D^{\otimes n} \to X^{\otimes n}$ by the rule

$$\theta\left((d_1 \otimes d_2 \otimes \cdots \otimes d_n) + N\right) = u(b_1 \otimes b_2 \otimes \cdots \otimes b_n) + v(c_1 \otimes c_2 \otimes \cdots \otimes c_n)$$

Where $d_1 \otimes d_2 \otimes \cdots \otimes d_n = ((b_1 \otimes b_2 \otimes \cdots \otimes b_n), (c_1 \otimes c_2 \otimes \cdots \otimes c_n))$.

It is easy to show that θ is well defined and also a homomorphism. Next we show the two triangles are commutative. For any $b_1 \otimes b_2 \otimes \cdots \otimes b_n \in B^{\otimes n}$, we have

$$\begin{aligned} &\theta t(b_1 \otimes b_2 \otimes \cdots \otimes b_n) \\ &= \theta\left((b_1 \otimes b_2 \otimes \cdots \otimes b_n), 0\right) + N \\ &= u(b_1 \otimes b_2 \otimes \cdots \otimes b_n) \end{aligned}$$

and for any $c_1 \otimes c_2 \otimes \cdots \otimes c_n \in C^{\otimes n}$, we have

$$\begin{aligned} &\theta g(c_1 \otimes c_2 \otimes \cdots \otimes c_n) \\ &= \theta\left(0, (c_1 \otimes c_2 \otimes \cdots \otimes c_n)\right) + N \\ &= v(c_1 \otimes c_2 \otimes \cdots \otimes c_n). \end{aligned}$$

So $\theta t = u$ and $\theta g = v$. Clearly θ is unique. This completes the proof. The following result is a well-known.

Theorem 6

The category $\mathcal{M}$ is complete. From Theorems 6, 7 and 8 we see that all the conditions of the Theorem 1 (Deleanu, 1975) are satisfied. So from the Theorem 1 (Deleanu et al., 1974) hence we have the following result.

Theorem 7

Every object $M^{\otimes n}$ of the category $\mathcal{M}$ has an Adams completion M_{S_n} with respect to the set of morphisms S_n. Furthermore, there exists a morphism $e : M^{\otimes n} \to M_{S_n}$ in $\bar{S}_n$ which is couniversal with respect to the morphisms in S_n: given a morphism $s : M^{\otimes n} \to N^{\otimes n}$ in S_n there exists a unique morphism $t : N^{\otimes n} \to M_{S_n}$ in $\bar{S}_n$ that $ts = e$. In other words the following diagram is commutative:

$$\begin{array}{ccc} M^{\otimes n} & \xrightarrow{e} & M_{S_n} \\ s \downarrow & \nearrow t & \\ N^{\otimes n} & & \end{array}$$

Theorem 8

The free K-module homomorphism $e : M^{\otimes n} \to M_{S_n}$ is in S_n.

Proof 5

Let $S_n^1 = \{f : M^{\otimes n} \to N^{\otimes n}$ in $\mathcal{M} \mid f$ is a subjective free K-module homomorphism$\}$

and

$S_n^2 = \{f : M^{\otimes n} \to N^{\otimes n}$ in $\mathcal{M} \mid f$ is a free K-module homomorphism $\}$.

Clearly,

(a) $S_n = S_n^1 \cap S_n^2$ and
(b) S_n^1 and S_n^2 satisfy all the conditions of Theorem 3.

Hence $e_n \in S_n$. This completes the proof.

Result

We show that the nth term of symmetric algebra $A^n(M)$ of a free K-module M, is precisely the Adams completion M_{S_n} of $M^{\otimes n}$.

Theorem 9

$A^n(M) \cong M_{S_n}$

Proof 6

Consider the following diagram:

$$\begin{array}{ccc} M^{\otimes n} & \xrightarrow{e} & M_{S_n} \\ \rho_M \downarrow & \swarrow \varphi & \\ A^n(M) & & \end{array}$$

By Theorem 5, there exists a unique morphism $\varphi : M_{S_n} \longrightarrow A^n(M)$ in S_n such that $\varphi e = \rho_M$.

Next consider the following diagram:

$M^{\otimes n} \quad \overset{e}{\rightarrow} \quad M_{S_n}$

$\rho_M \downarrow \quad \nearrow \psi$

$A^n(M)$

By Theorem 7, there exists a unique morphism $\psi: A^n(M) \longrightarrow M_{S_n}$ in S_n such that $\psi\rho_M = e$. Consider the following diagram:

$M^{\otimes n} \quad \overset{e}{\rightarrow} \quad M_{S_n}$

$\nearrow \psi$

$e \downarrow \quad A^n(M)$

$\nearrow \varphi \quad \nearrow 1_{M_{S_n}}$

M_{S_n}

We have $\psi\varphi e = \psi\rho_M = e$. By the uniqueness condition of the co-universal property of e, we conclude that $\psi\varphi = 1_{M_{S_n}}$. Next consider the following diagram:

$M^{\otimes n} \quad \overset{\rho_M}{\rightarrow} \quad A^n(M)$

$\nearrow \varphi$

$\rho_M \downarrow \quad M_{S_n}$

$\nearrow \psi \quad \nearrow 1_{A^n(M)}$

$A^n(M$

We have $\varphi\psi\rho_M = \varphi e = \rho_M$. By the uniqueness condition of the property of ρ_M, we conclude that $\varphi\psi = 1_{A^n(M)}$. Thus $A^n(M) \cong M_{S_n}$. This completes the proof.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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