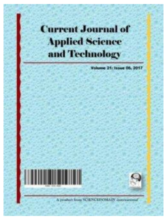

Current Journal of Applied Science and Technology

28(3): 1-5, 2018; Article no.CJAST.42693
ISSN: 2457-1024
(Past name: British Journal of Applied Science & Technology, Past ISSN: 2231-0843, NLM ID: 101664541)

A Study on Two Special Ternary Quadratic Diophantine Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Author SA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author MAG managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/CJAST/2018/42693
Editor(s):
(1) Dr. Qing-Wen Wang, Professor, Department of Mathematics, Shanghai University, P.R. China.
Reviewers:
(1) Tunjo Perić, University of Zagreb, Croatia.
(2) Teodoro Lara, University of Los Andes, Venezuela.
Complete Peer review History: http://www.sciencedomain.org/review-history/25593

Short Research Article

Received 5th May 2018
Accepted 10th July 2018
Published 18th July 2018

ABSTRACT

Hyperbolic paraboloids represented by the ternary quadratic diophantine equations given by $(k+1)^2 x^2 - k^2 y^2 = 2z$ and $k^2 a y^2 - (a-k+1)x^2 = ((k^2-1)a + k - 1)z$, $a > k-1 > 0$ are respectively considered. Employing matrix method, generation formula for integer solutions to each of the above hyperbolic paraboloids is constructed in the present study.

Keywords: Ternary quadratic; non-homogeneous quadratic; generation of solutions; hyperbolic paraboloid.

1. INTRODUCTION

One of the oldest and largest branches of Number theory is the subject of diophantine equations which has been considered by many mathematicians since antiquity. The study of diophantine equations is the study of solutions of polynomial equations or systems of equations in

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integers, rational numbers or sometimes more general number rings. A diophantine equation is an indeterminate polynomial equation that allows the variables to be integers. A natural extension of ordinary integers into complex numbers is the Gaussian integers in which, both the real and imaginary parts are integers. It is quite obvious that diophantine equations are rich in variety and several methods are available to obtain solutions either in real integers or in Gaussian integers. In this context, one may refer findings of other researchers [1-10].

The aim of this research work is to illustrate a process for generating the sequence of integer solutions to the given hyperbolic paraboloid based on its known solution.

2. METHOD OF ANALYSIS

2.1 Hyperbolic Paraboloid: 1

Consider the hyperbolic paraboloid given by:

$$(k+1)^2 x^2 - k^2 y^2 = 2z \quad (1)$$

Introduction of the linear transformations

$$x = X + k^2 T \ , \ y = X + (k+1)^2 T \ , \ z = (2k+1)W \quad (2)$$

leads to

$$X^2 = k^2 (k+1)^2 T^2 + 2W$$

Which is satisfied by

$$T = 4K \ , W = 2(2k+1)^2 K^2 \Rightarrow X = 2K\left(2k^2 + 2k + 1\right)$$

In view of (2), we have

$$x = \left(8k^2 + 4k + 2\right)K \ , \ y = \left(8k^2 + 12k + 6\right)K \ , \ z = 2\left(2k+1\right)^3 K^2 \quad (3)$$

Denote the above values of x, y, z as x_0, y_0, z_0 respectively. A process of obtaining a sequence of integer solutions to the given equation based on its given solution has been illustrated (3).

Let (x_1, y_1, z_1) given as:

$$x_1 = h - (2k+1)x_0 \ , \ y_1 = h + (2k+1)y_0 \ , \ z_1 = (2k+1)^2 z_0 \quad (4)$$

which is also a solution of (1) as well in which h is an unknown to be determined.

Substitution of (4) in (1) gives the value of h to be

$$h = 2(k+1)^2 x_0 + 2k^2 y_0 \quad (5)$$

Using (5) in (4), the second solution is given by

$$x_1 = \left(2k^2 + 2k + 1\right)x_0 + 2k^2 y_0 \ , \ y_1 = \left(2k^2 + 4k + 2\right)x_0 + \left(2k^2 + 2k + 1\right)y_0 \quad (6)$$

and $\quad z_1 = (2k+1)^2 \ z_0 \quad (7)$

Proposition:

The n^{th} solution (x_n, y_n, z_n) of (1) is represented by

$$x_n = h - (2k+1)x_{n-1} \ , \ y_n = h + (2k+1)y_{n-1} \ , \ z_n = (2k+1)^2 z_{n-1}$$

Proof:

To obtain the values of x_n, y_n, the following steps have been taken.

The solution (6) is written in the matrix form as follows:

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

Where, t is the transpose and M is the matrix of order 2 x 2 given by

$$M = \begin{bmatrix} 2k^2 + 2k + 1 & 2k^2 \\ 2k^2 + 4k + 2 & 2k^2 + 2k + 1 \end{bmatrix}$$

In general, $(x_n, y_n)^t = M^n (x_0, y_0)^t$

To find M^n, consider the characteristic equation

$$|M - \lambda I| = 0 \quad (8)$$

Where, I is the unit matrix of order 2 and λ is the Eigen value of M.

Solving (8), the Eigen values of M are given by $\lambda_1 = (2k+1)^2 \ , \ \lambda_2 = 1$.

It is well-known that,

$$M^n = \frac{\lambda_1^n}{\lambda_1 - \lambda_2}(M - \lambda_2 I) + \frac{\lambda_2^n}{\lambda_2 - \lambda_1}(M - \lambda_1 I)$$

$$= \frac{1}{(4k^2+4k)}\begin{bmatrix} (2k^2+2k)\left((2k+1)^{2n}+1\right) & 2k^2\left((2k+1)^{2n}-1\right) \\ (2k^2+4k+2)\left((2k+1)^{2n}-1\right) & (2k^2+2k)\left((2k+1)^{2n}+1\right) \end{bmatrix}$$

Thus, we have

$$\left.\begin{aligned} x_n &= \frac{1}{(4k^2+4k)}\left[(2k^2+2k)\left((2k+1)^{2n}+1\right)x_0 + 2k^2\left((2k+1)^{2n}-1\right)y_0\right] \\ y_n &= \frac{1}{(4k^2+4k)}\left[(2k^2+4k+2)\left((2k+1)^{2n}-1\right)x_0 + (2k^2+2k)\left((2k+1)^{2n}+1\right)y_0\right] \end{aligned}\right\} \quad (9)$$

$$z_n = (2k+1)^{2n} z_0 \quad (10)$$

Thus, (9) and (10) represent the generation formula for the given hyperbolic paraboloid in terms of its given solution.

2.2 Hyperbolic Paraboloid: 2

Consider the hyperbolic paraboloid given by

$$k^2ay^2 - (a-k+1)x^2 = ((k^2-1)a+k-1)z,\ a > k-1 > 0 \quad (1)$$

Introduction of the linear transformations

$$x = X + k^2aT\ ,\ y = X + (a-k+1)T \quad (2)$$

leads to

$$X^2 = k^2a(a-k+1)T^2 + z$$

which is satisfied by

$$T = K\ ,\ z = \left(\frac{(k-1)k}{2}\right)^2 K^2\ \Rightarrow X = K\left(ak - \frac{(k-1)k}{2}\right)$$

In view of (2), we have

$$x = \frac{K}{2}\left(2a(k^2+k) - k(k-1)\right),\ y = \frac{K}{2}\left(2a(k+1) - (k-1)(k+2)\right) \quad (3)$$

Denote the above values of x, y, z as x_0, y_0, z_0 respectively. A process of obtaining sequence of integer solutions to the given equation based on its given solution has been illustrated in the present study (3).

Let, (x_1, y_1, z_1) given as:

$$x_1 = h + \left((k^2-1)a+k-1\right)x_0\ ,\ y_1 = h - \left((k^2-1)a+k-1\right)y_0\ ,$$
$$z_1 = \left((k^2-1)a+(k-1)\right)^2 z_0 \quad (4)$$

It is also a solution of (1) as well in which h is an unknown to be determined.

Substitution of (4) in (1) gives the value of h to be

$$h = (2a-2k+2)x_0 + 2k^2ay_0 \quad (5)$$

Using (5) in (4), the second solution is given by

$$x_1 = \left((k^2+1)a-k+1\right)x_0 + 2k^2a\,y_0\ ,\ y_1 = (2a-2k+2)x_0 + \left((k^2+1)a-k+1\right)y_0 \quad (6)$$

and $z_1 = \left((k^2-1)a+(k-1)\right)^2 z_0$ (7)

Proposition:

The n^{th} solution (x_n, y_n, z_n) of (1) is represented by

$$x_n = h + \left((k^2-1)a+k-1\right)x_{n-1}\ ,\ y_n = h - \left((k^2-1)a+k-1\right)y_{n-1}\ ,$$
$$z_n = \left((k^2-1)a+(k-1)\right)^2 z_{n-1}$$

Proof:

To obtain the values of x_n, y_n, the following steps have been taken.

The solution (6) is written in the matrix form as:

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

Where, t is the transpose and M is the matrix of order 2 x 2 given by

$$M = \begin{bmatrix} (k^2+1)a - k + 1 & 2k^2 a \\ 2a - 2k + 2 & (k^2+1)a - k + 1 \end{bmatrix}$$

In general, $(x_n, y_n)^t = M^n (x_0, y_0)^t$

To find M^n, consider the characteristic equation

$$|M - \lambda I| = 0 \tag{8}$$

Where, I is the unit matrix of order 2 and λ is the Eigen value of M

Solving (8), the Eigen values of M are given by

$$\lambda_1 = (k^2+1)a - k + 1 + 2k\sqrt{a^2 - ak + a}\,,$$
$$\lambda_2 = (k^2+1)a - k + 1 - 2k\sqrt{a^2 - ak + a}$$

It is well-known that

$$M^n = \frac{\lambda_1^n}{\lambda_1 - \lambda_2}(M - \lambda_2 I) + \frac{\lambda_2^n}{\lambda_2 - \lambda_1}(M - \lambda_1 I)$$
$$= \frac{1}{4k\sqrt{a^2 - ak + a}} \begin{bmatrix} 2k\sqrt{a^2 - ak + a}\,(\alpha^n + \beta^n) & 2k^2 a(\alpha^n - \beta^n) \\ (2a - 2k + 2)(\alpha^n - \beta^n) & 2k\sqrt{a^2 - ak + a}\,(\alpha^n + \beta^n) \end{bmatrix}$$

Thus, we have

$$\left.\begin{aligned} x_n &= \frac{1}{4k\sqrt{a^2 - ak + a}}\left[2k\sqrt{a^2 - ak + a}\,(\alpha^n + \beta^n)x_0 + 2k^2 a(\alpha^n - \beta^n)y_0\right] \\ y_n &= \frac{1}{4k\sqrt{a^2 - ak + a}}\left[(2a - 2k + 2)(\alpha^n - \beta^n)x_0 + 2k\sqrt{a^2 - ak + a}\,(\alpha^n + \beta^n)y_0\right] \end{aligned}\right\} \tag{9}$$

$$z_n = \left((k^2 - 1)a + (k - 1)\right)^{2n} z_0 \tag{10}$$

Thus, (9) and (10) represent the generation formula for the given hyperbolic paraboloid in terms of its given solution.

3. CONCLUSION

From the present study, a general formula generating sequence of solutions to the given equations based on its initial solution has been successfully obtained. As the diophantine equations are rich in variety due to its definition, therefore, one may attempt for obtaining generation formula for other choices of hyperbolic paraboloid.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history/25593