Asian Research Journal of Mathematics

7(4): 1-8, 2017; Article no.ARJOM.38546

ISSN: 2456-477X

Exploring Some Statistical Properties of the Concomitants of Upper Record Statistics for Bivariate Pseudo-Rayleigh Distribution

Hesham M. Reyad[1*] and Soha A. Othman[2]

[1]*College of Business and Economics, Qassim University, Kingdom of Saudi Arabia.*
[2]*Institute of Statistical Studies and Research, Cairo University, Egypt.*

Authors' contributions

This work was carried out in collaboration among both authors. Author HMR introduced the idea in a methodically structure, did the data analysis and drafted the manuscript. Author SAO assisted in building the study design and also did the final proofreading. The two authors managed the analyses of the study and literature searches and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/38546
Editor(s):
(1) Jitender Singh, Guru Nanak Dev University, Punjab, India.
Reviewers:
(1) Rashmi Awad, Devi Ahilya University, India.
(2) Derek Young, University of Kentucky, USA.
Complete Peer review History: http://www.sciencedomain.org/review-history/22267

Original Research Article

Received: 30th November 2017
Accepted: 8th December 2017
Published: 13th December 2017

Abstract

In this paper, we have obtained the distribution of the kth concomitant and the joint distribution of kth and mth concomitants of upper record statistics for the bivariate pseudo-Rayleigh distribution. Some statistical properties for the resulting distributions are also discussed such as; single and product moments, percentile, reliability and hazard functions and moments generating functions.

Keywords: Concomitants; record statistics; pseudo-Rayleigh distribution.

1 Introduction

The study of record values and associated statistics is popular and important in many real life applications, such as: education, weather, economic, sports data and analysis of lifetime data. The idea of record values

*Corresponding author: E-mail: hesham_reyad@yahoo.com;

and many of their basic properties were first emerged by Chandler [1]. Numerous studies on records were publishes throughout the 1950s, for example; Foster and Stuart [2] have introduced the first idea of nonparametric inference from records. Rensick [3] and Shorrok [4] explained the asymptotic theory of records. For properties of record values refer to Ahsanullah [5], Arnold et al. [6] and Balakrishnan and Balasubramanian [7].

A number of statisticians have studied inference based on record samples for certain distributions. Ahsanullah [8] has obtained some statistical properties of records for the Lomax distribution. Balakrishnan et al. [9] have established some recurrence relations for moments of record values for Gumbel distribution. For more studies refer to Nevzorov [10], Kamps [11], Raqab [12] and Sultan [13].

The distribution of the kth upper record, $X_{U(k)}$, was defined by Ahsanullah [14] as:

$$f_{k:n}(x_k) = \frac{1}{\Gamma(k)} f(x_k)\left[R(x_k)\right]^{k-1}, \qquad -\infty < x_k < \infty \tag{1.1}$$

where $R(x) = -\ln\left[1 - F(x)\right]$ and $F(x)$ is the distribution function of X.

Ahsanullah [14] has further shown that the joint distribution of kth and mth upper records, $X_{U(k)}$ and $X_{U(m)}$ for $k < m$ is given by:

$$\begin{aligned} f_{k,m:n}(x_k, x_m) = & \frac{1}{\Gamma(k)\Gamma(m-k-1)} r(x_k) f(x_m) \left[R(x_k)\right]^{k-1} \\ & \times\left[R(x_m) - R(x_k)\right]^{m-k-1} \qquad ,-\infty < x_k < x_m < \infty \end{aligned} \tag{1.2}$$

where $r(x) = R'(x)$.

Let $(X_i, Y_i), i = 1, 2, ..., n$ be n independent random variables from some bivariate distribution. If we arrange the X-variates in ascending order as $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ then the Y-variates paired with these order statistics are denoted by $Y_{[1:n]}, Y_{[2:n]}, ...Y_{[n:n]}$ and termed the concomitants of order statistics (Yang [15]). A few studies concerned with the concomitants of record values. Ahsanullah et al. [16] has studied the concomitants of upper record statistics for bivariate pseudo-Weibull distribution. Mohsin et al. [17] has investigated some statistical properties of the concomitants of record statistics for bivariate pseudo-Exponential distribution.

The distribution of kth concomitant of record values is defined by Ahsanullah [14] as:

$$f(y_k) = \int_{-\infty}^{\infty} f(y|x) f_{k:n}(x_k)\, dx \tag{1.3}$$

where $f_{k:n}(x_k)$ is the distribution of kth upper record given in (1.1) and $f(y|x)$ is the conditional distribution of Y given $X = x$. The joint distribution of kth and mth concomitants of upper record values is also given by Ahsanullah [14] as:

$$f(y_k, y_m) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_m} f(y_k|x_k) f(y_m|x_m) f_{k,m:n}(x_k, x_m)\, dx_k\, dx_m \tag{1.4}$$

where $f_{k,m:n}(x_k, x_m)$ is the joint distribution of kth and mth upper records given in (1.2).

In this paper, we will investigate the distributions of the concomitants of upper record statistics for the bivariate pseudo-Rayleigh distribution. In Section 2, we gave a brief review for the bivariate pseudo-Rayleigh distribution introduced by Shahbaz and Shahbaz [18]. In Section 3, we have derived the distribution of kth concomitant of upper record statistics for bivariate pseudo-Rayleigh distribution with some basic properties. The joint distribution of kth and mth concomitants of upper record statistics for the bivariate pseudo-Rayleigh distribution and some characterizations are given in Section 4. Some concluding remarks are given in the last Section.

2 Bivariate Pseudo-Rayleigh Distribution

The pseudo distribution is a new category of multivariate distributions introduced by Filus and Filus [19] as a joint distribution of linear combinations of independent random variables. Many authors have investigated the pseudo distributions; for examples: Filus and Filus [19] have presented the pseudo-Weibull and pseudo-Gamma distributions. Hanif [20] has studied the pseudo-Gaussian and pseudo-Weibull distributions. Shahbaz et al. [21] has introduced the bivariate pseudo-Exponential distribution. In this paper, we consider the bivariate pseudo-Raleigh distribution introduced by Shahbaz and Shahbaz [18] as:

$$f(x,y) = 4\alpha x^3 y e^{-x^2(\alpha+y^2)},\ x,y,\alpha > 0 \tag{2.1}$$

The marginal distribution of X is:

$$f(x) = 2\alpha x e^{-\alpha x^2},\ x,\alpha > 0 \tag{2.2}$$

The distribution function of X is:

$$F(x) = 1 - e^{-\alpha x^2},\ x,\alpha > 0 \tag{2.3}$$

The conditional distribution of Y given $X = x$ is:

$$f(y|x) = 2x^2 y e^{-x^2 y^2},\ x,y,\alpha > 0 \tag{2.4}$$

3 Distribution of kth Concomitant of Upper Record Values and Its Properties

In this section, we have obtained the distribution of kth concomitant of upper record statistics for the bivariate pseudo-Rayleigh distribution given in (2.1) and also derived some corresponding statistical properties.

The distribution of kth concomitant of record statistics for (2.1) can be derived by using (1.3). To obtain the distribution (1.3) we first find the distribution of kth upper record given in (1.1). The distribution (1.1) is given as:

$$f_{k:n}(x_k) = \frac{2\alpha^k}{\Gamma(k)} x^{2k-1} e^{-\alpha x^2} \tag{3.1}$$

Now using (2.4) and (3.1) in (1.3), the distribution of kth concomitant of upper record statistics for the bivariate pseudo-Rayleigh distribution is given as below:

$$f(y_k)=\frac{4\alpha^k y}{\Gamma(k)}\int_0^\infty x^{2k+1}\,e^{-x^2(\alpha+y^2)}\,dx=2k\alpha^k y\,(\alpha+y^2)^{-(k+1)},\ \ y,\alpha,k>0 \tag{3.2}$$

The distribution function of (3.2) is:

$$F(y)=1-\alpha^k(\alpha+y^2)^{-k},\qquad y,\alpha,k>0 \tag{3.3}$$

Using (3.3), the reliability function of (3.2) is:

$$R(y)=\alpha^k(\alpha+y^2)^{-k},\qquad y,\alpha,k>0 \tag{3.4}$$

Moreover, the hazard function can be obtained by using (3.2) and (3.4) to be:

$$h(y)=\frac{2k\,y}{\alpha+y^2},\qquad y,\alpha,k>0 \tag{3.5}$$

The rth moments of the distribution (3.2) is derived as:

$$\mu_r'=E(y^r)=2k\alpha^k\int_0^\infty y^{r+1}\,(\alpha+y^2)^{-(k+1)}\,dy$$

or

$$\mu_r'=\frac{\alpha^{r/2}\,\Gamma(k-r/2)\,\Gamma(r/2+1)}{\Gamma(k)},\quad k>\frac{r}{2} \tag{3.6}$$

Consequently, the mean and variance of (3.2) are given as:

$$E(Y)=\frac{\pi\ \sqrt{\alpha}\ \Gamma(k-1/2)}{2\,\Gamma(k)}\quad\text{and}\quad \mathrm{var}(Y)=\frac{\alpha\left[4\Gamma^2(k)-(k-1)\,\pi^2\,\Gamma^2(k-1/2)\right]}{4(k-1)\Gamma^2(k)} \tag{3.7}$$

The coefficients of skewness and kurtosis of (3.2) are:

$$\xi_1=\frac{3\pi\,(k-1)^{3/2}\,\Gamma(k-3/2)}{4\Gamma(k)}\quad\text{and}\quad \xi_2=\frac{2(k-1)}{k-2} \tag{3.8}$$

The pth percentile of (3.2) can be obtained from (3.3) as:

$$\delta_p=\left[\frac{\alpha\left[1-(1-p)^{1/k}\right]}{(1-p)^{1/k}}\right]^{1/2} \tag{3.9}$$

The median of (3.2) can be calculated by using $p=0.5$ in (3.9).

The mode of (3.2) can be derived to be:

$$\omega=\left(\frac{\alpha}{2k+1}\right)^{1/2} \tag{3.10}$$

The moment generating function of (3.2) can be obtained as:

$$M_y(t) = E(e^{ty}) = 2k\alpha^k \int_0^\infty e^{ty} y\ (\alpha + y^2)^{-(k+1)} dy$$

or

$$M_y(t) = \sum_{j=0}^{\infty} \frac{\alpha^{j/2}\, t^j\, \Gamma(k - j/2)\Gamma(j/2+1)}{\Gamma(j+1)\Gamma(k)} \tag{3.11}$$

4 Distribution of kth and mth Concomitant of Upper Record Values

In this section, we have derived the joint distribution of kth and mth concomitants of upper record statistics for the bivariate pseudo-Rayleigh distribution given in (2.1). Furthermore, expressions of product moments and joint moment generating function for the resulting distribution are also obtained.

The joint distribution of kth and mth concomitants of upper record statistics can be obtained by using (1.4). To derive the distribution (1.4) we first deduct the joint distribution of two upper record statistics $X_1 = X_{U(k)}$ and $X_2 = X_{U(m)}$ given in (1.2). The distribution (1.2) is given as:

$$f_{k,m:n}(x_k, x_m) = \frac{4\alpha^m}{\Gamma(k)\Gamma(m-k-1)} x_1^{2k-1}\, x_2 \left[x_2^2 - x_1^2\right]^{m-k-1} e^{-\alpha x_2^2} \tag{4.1}$$

By using (2.4) and (4.1) in (1.4), the joint distribution of kth and mth concomitants of record values for bivariate pseudo-Rayleigh distribution is given as follows:

$$\begin{aligned} f(y_1, y_2) &= \frac{16\alpha^m}{\Gamma(k)\Gamma(m-k-1)} \int_0^\infty \int_0^{x_2} y_1\, y_2\, x_1^{2k+1}\, x_2^3 \left[x_2^2 - x_1^2\right]^{m-k-1} \mathrm{e}^{-x_1^2 y_1^2}\ \mathrm{e}^{-x_2^2(\alpha + y_2^2)}\, dx_1\, dx_2 \\ &= \frac{16\alpha^m y_1 y_2}{\Gamma(k)\Gamma(m-k-1)} \left[\int_0^\infty x_2^3\, \mathrm{e}^{-x_2^2(\alpha+y_2^2)}\, dx_2\right]\left[\int_0^{x_2} x_1^{2k+1}\left[x_2^2 - x_1^2\right]^{m-k-1} \mathrm{e}^{-x_1^2 y_1^2}\, dx_1\right] \end{aligned}$$

or

$$f(y_1, y_2) = \frac{16\alpha^m}{\Gamma(k)\Gamma(m-k-1)} \int_0^\infty x_2^3\, \mathrm{e}^{-x_2^2(\alpha+y_2^2)}\, A(x_2)\, dx_2 \tag{4.2}$$

where

$$A(x_2) = \int_0^{x_2} x_1^{2k+1}\left[x_2^2 - x_1^2\right]^{m-k-1} \mathrm{e}^{-x_1^2 y_1^2}\, dx_1 \tag{4.3}$$

Integrating (4.3), we have

$$A(x_2) = \sum_{s=0}^{\infty} \sum_{h=0}^{m-k-1} \binom{m-k-1}{h} (-1)^{h+s} \frac{y_1^{2s}\, x_2^{2(m+s)}}{2(k+h+s+1)\Gamma(s+1)} \tag{4.4}$$

By using (4.4) in (4.2) and after simplifications we obtain

$$f(y_1,y_2)=\frac{4\alpha^m}{\Gamma(k)\Gamma(m-k-1)}\sum_{s=0}^{\infty}\sum_{h=0}^{m-k-1}\binom{m-k-1}{h}(-1)^{h+s}$$
$$\times\frac{\Gamma(m+s+2)}{(k+h+s+1)\Gamma(s+1)}y_1^{2s+1}\,y_2(\alpha+y_2^2)^{-(m+s+2)} \tag{4.5}$$

where $Y_1=Y_k$ and $Y_2=Y_m$

The product moments for (4.5) are derived as:

$$\mu'_{q,r}=E\left(y_1^q\,y_2^r\right)=\int_0^\infty\int_0^\infty y_1^q\,y_2^r\,f(y_1,y_2)\,dy_1\,dy_2 \tag{4.6}$$

Putting the values of $f\,(y_1,y_2)$ from (4.5) in (4.6), we have

$$\mu'_{q,r}=\frac{4\alpha^m}{\Gamma(k)\Gamma(m-k-1)}\sum_{s=0}^{\infty}\sum_{h=0}^{m-k-1}\binom{m-k-1}{h}(-1)^{h+s}\,\frac{\Gamma(m+s+2)}{(k+h+s+1)\Gamma(s+1)}$$
$$\times\int_0^\infty\int_0^\infty y_1^{q+2s+1}\,y_2^{r+1}\,(\alpha+y_2^2)^{-(m+s+2)}\,dy_1\,dy_2$$

After some algebra, we obtain:

$$\mu'_{q,r}=\frac{2}{\Gamma(k)\Gamma(m-k-1)}\sum_{s=0}^{\infty}\sum_{h=0}^{m-k-1}(-1)^{s+h+\ell}\,\alpha^{r/2-s-1}\sum_{\ell=0}^{q+2s+1}\binom{m-k-1}{h}\binom{q+2s+1}{\ell}$$
$$\times\frac{\alpha^{r/2-s-1}\,\Gamma(m+s-r/2-1)\,\Gamma(r/2+1)}{(k+h+s+1)(\ell-q-2s-2)\Gamma(s+1)} \tag{4.7}$$

The joint moment generating functions of (4.5) is given by:

$$M_{y_1,y_2}(t_1,t_2)=E\left(e^{t_1y_1+t_2y_2}\right)=\int_0^\infty\int_0^\infty e^{t_1y_1+t_2y_2}\,f(y_1,y_2)\,dy_1\,dy_2 \tag{4.8}$$

Using (4.5) in (4.8), we get

$$M_{y_1,y_2}(t_1,t_2)=\frac{4\alpha^m}{\Gamma(k)\Gamma(m-k-1)}\sum_{s=0}^{\infty}\sum_{h=0}^{m-k-1}\binom{m-k-1}{h}(-1)^{h+s}\,\frac{\Gamma(m+s+2)}{(k+h+s+1)\Gamma(s+1)}$$
$$\times\int_0^\infty\int_0^\infty e^{t_1y_1+t_2y_2}\,y_1^{2s+1}y_2(\alpha+y_2^2)^{-(m+s+2)}\,dy_1\,dy_2$$

After simplifications, we obtain

$$M_{y_1,y_2}(t_1,t_2)=\frac{2}{\Gamma(k)\Gamma(m-k-1)}\sum_{s=0}^{\infty}\sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\sum_{h=0}^{m-k-1}\sum_{d=0}^{2s+w_1+1}\binom{m-k-1}{h}\binom{2s+w_1+1}{d}$$
$$\times\frac{(-1)^{h+s+d}\,\alpha^{\frac{w_2}{2}-s-1}\,t_1^{w_1}\,t_2^{w_2}\,\Gamma(m+s-w_2/2+1)\,\Gamma(w_2/2+1)}{(k+h+s+1)(d-2s-w_1-2)\Gamma(\mathrm{s}+1)\,\Gamma(w_1+1)\,\Gamma(w_2+1)} \tag{4.9}$$

5 Conclusion

We introduced and studied the distribution of the kth concomitant and the joint distribution of kth and mth concomitants of upper record statistics for the bivariate pseudo-Rayleigh distribution. Some mathematical properties for these distributions are derived such as; single and product moments, percentile, reliability and hazard functions and moments generating functions.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Chandler KN. The distribution and frequency of record values. J. Royal Statist. Soc. 1952;14:220-228.

[2] Foster FG, Stuart A. Distribution free tests in time series based on braking of records. Journal of Royal Statistics Society. 1954;16:1-22.

[3] Resnick SI. Limit laws of record values. J. Stochastic Processes Appl. 1973;1:67-82.

[4] Shorrock RW. Record values and inter-record times. J. Appl. Prob. 1973;10(3):543-555.

[5] Ahsanullah M. Characterization of exponential distribution by record values. Sankhya. 1979;B,41: 116-121.

[6] Arnold BC, Balakrishnan N, Nagaraja H. A first course in order statistics. John Wiley, New York; 1992.

[7] Balakrishnan N, Balakrishnan K. A characterization of Geometric distribution based on record values. J. Appl. Statist. Science. 1995;2(1):73-87.

[8] Ahsanullah M. Record values of Lomax distribution. Statist. Nederlandica. 1991;41(1):21-29.

[9] Balakrishnan N, Ahsanullah M, Chen PS. Relations for single and product moments of record values for Gumbel distribution. Stat. and Prob. Lett. 1995;15(3):223-227.

[10] Nevzorov VB. Record: Mathematical theory. Translations of mathematical monographs. American Mathematical Society. 2001;194.

[11] Kamps U. A concept of generalized order statistics. J. Statist. Plann. Inference. 1995;48:1-23.

[12] Raqab MZ, Ahsanullah M. On moment generating function of record from extreme value distribution. Pak. J. Statist. 2003;19(1):1-13.

[13] Sultan KS. Moments of record values from uniform distribution and associated inference. Egypt. Statist. J. ISSR, UNIV. 2002;44121:137-149.

[14] Ahsanullah M. Record statistics. Nova Science Publishers, Inc; New York; 1995.

[15] Yang SS. General distribution theory of concomitants of order statistics. Ann. Statist. 1977;5:996-1002.

[16] Ahsanullah M, Shahbaz S, Shahbaz MQ, Mohsin M. Concomitants of upper record statistics for bivariate pseudo-Weibull distribution. Applications and Applied Mathematics; an International Journal. 2010;5(2):282-291.

[17] Mohsin M, Pliz J, Gunter S, Shahbaz S, Shahbaz M. Some distributional properties of the concomitants of record statistics for bivariate pseudo-Exponential distribution and characterization. Journal of Prime Research in Mathematics. 2010;6:32-37.

[18] Shahbaz MQ, Shahbaz S. Order statistics and concomitants of bivariate pseudo-Rayleigh distribution. World App. Sci. J. 2009;7(7):826-828.

[19] Filus JK, Filus LZ. On some new classes of multivariate probability distributions. Pak. J. Statist. 2006;22(1): 21-42.

[20] Hanif S. Concomitants of ordered random variables. Unpublished PHD Thesis, NCBA & E; 2007.

[21] Shahbaz S, Shahbaz MQ, Mohsin M. On concomitants of order statistics for bivariate pseudo-exponential distribution. World App. Sci. J. 2009;6(8):1151-1156.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/22267