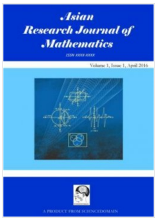

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Conditional Least Squares Estimation for Discretely Sampled Nonergodic Diffusions

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

Strong consistency and conditional asymptotic normality of the conditional least squares estimator of a parameter appearing nonlinearly in the time dependent drift coefficient of the Itô stochastic differential equation are obtained under some regularity conditions when the corresponding diffusion is observed at discretely spaced dense time points satisfying a moderately increasing experimental design condition, the case of high frequency data. Main results are illustrated by the mean reversion process with drift and the nonhomogeneous Ornstein-Uhlenbeck process.

Keywords: Nonhomogeneous Itô stochastic differential equation; nonergodic diffusion process; discrete observations.

Mathematics Subject Classification(2010): 60H10, 60J60, 62M05, 62F12.

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1 Introduction

Time dependent diffusion models are useful for modeling term structure dynamics in finance, see [1]. Parameter estimation for continuously observed diffusions is now classical, see e.g., [2, 3, 4, 5, 6, 7, 8, 9]. When the parameter space is bounded, [10, 5, 6, 7] studied the asymptotic properties of maximum likelihood and Bayes estimators of the drift parameter as the intensity of noise $\epsilon \to 0$ and also as the time horizon $T \to \infty$. When the parameter space is unbounded, [11, 12] studied the asymptotic properties of MLE as $T \to \infty$.

However, it is difficult to observe the diffusion continuously. In view of this and applications to finance, parameter estimation for discretely observed diffusions is the recent trend of investigation. Over the last three decades, several methods of parameter estimation in stationary homogeneous diffusions have been studied by [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. Beyond the first order asymptotics, [50] obtained large deviations probability and Berry-Esseen type bounds for approximate maximum likelihood estimators for homogeneous Ornstein-Uhlenbeck process. [51] studied Berry-Esseen type bounds for the approximate minimum contrast estimators. See [36] for results on higher order likelihood asymptotics and Bayesian asymptotics for drift estimation of finite and infinite dimensional stochastic differential equations.

Many SDEs applied to finance are nonhomogeneous, nonstationary and nonergodic. Stationarity and mixing seems to be a restriction on many financial applications. For instance, the instantaneous return and price volatility change over time and price level. For homogeneous but nonergodic diffusions, in [52] Ait-Sahalia studied asymptotic behavior of maximum likelihood estimator by obtaining a closed form approximation of the likelihood using Hermite expansion. Parameter estimation for discretely observed nonhomogeneous diffusion process has been paid some attention. In [53, 54, 55], Pedersen studied the asymptotic properties of importance sampling type approximate maximum likelihood estimator based on approximation of the transition probability density of the discretely observed diffusion using Euler scheme. In [34], Elerian, Chib and Shephard used MCMC method. In [56], Harison obtained the consistency and the asymptotic normality of the maximum contrast estimator of a nonlinear parameter in a linear nonhomogeneous SDE. In the likelihood based approach, it is difficult to calculate the transition probability density of the discretized process. To avoid this problem, we adopt conditional least squares approach. Also, conditional least squares estimator is simple to compute for high frequency data and it corresponds to the Euler scheme for discretization of the stochastic differential equations, see [57]. Here we obtain strong consistency and conditional asymptotic normality of the conditional least squares estimator (CLSE) of a parameter appearing nonlinearly in the drift coefficient of a nonhomogeneous SDE under some regularity conditions. Our arguments are related to the statistical inference for nonergodic models, see e.g., [58, 59].

For conditional asymptotic normality we do not need the RIED (rapidly increasing experimental design) condition i.e. $T \to \infty$ and $\frac{T}{\sqrt{n}} \to 0$. This condition was used by Prakasa Rao [14] for the stationary homogeneous SDE. We weaken the design condition which we call the *moderately increasing experimental design* , i.e., $T \to \infty$ and $\frac{T}{n^{2/3}} \to 0$ to obtain asymptotic normality.

The organization of the paper is as follows : Section 2 describes the model and the assumptions. Section 3 contains the strong consistency of the CLSE and Section 4 contains conditional asymptotic normality of the CLSE. Section 5 gives two examples from non-homogeneous diffusion process where the results of the previous sections apply.

2 Model and Assumptions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis satisfying the usual hypotheses on which is defined a real valued diffusion process satisfying the one-dimensional Itô SDE

$$dX_t = f(\theta, t, X_t)\ dt\ +\ dW_t,\quad t \geq 0,\quad X_0 = \xi_0 \tag{2.1}$$

where $\{W_t\}$ is a standard Wiener process adapted to the filtration $\{\mathcal{F}_t\}$ such that for $0 \leq s \leq t, W_t - W_s$ is independent of $\mathcal{F}_s, f$ is a known real valued continuous (in θ) function defined on $\Theta \times [0,T] \times \mathbb{R}$, where Θ is a compact subset of the real line which contains the unknown parameter θ and $E[\xi_0^2] < \infty$. Let θ_0 be the true value of the parameter, which lies inside the parameter space Θ.

Suppose the process $\{X_t\}$ is observed at known real time points $t_k = k\frac{T}{n}, k = 0, 1, 2, \ldots, n$ where $\frac{T}{n} \to 0$ and $T \to \infty$. For simplicity only we take equally spaced time interval. One could take unequally spaced time interval with mesh of the partition going to zero. We estimate θ based on these observations. Let

$$Q_{n,T}(\theta) := \frac{n}{T}\sum_{k=0}^{n-1}\left[X_{t_{k+1}} - X_{t_k} - f(\theta, t_k, X_{t_k})\frac{T}{n}\right]^2 \tag{2.2}$$

The conditional least squares estimator (CLSE) of θ is defined as

$$\theta_{n,T} := \arg\inf_{\theta\in\Theta} Q_{n,T}(\theta).$$

Since Θ is compact and f is continuous in θ, there exists a measurable CLSE by using Lemma 7.3.3 of [60, p. 307]. Henceforth, we will always assume the existence of such a measurable CLSE. Note that this estimator coincides with discretized Euler estimator for nonhomogeneous diffusion.

Let P_θ^T be the measure generated by the process $\{X_t, 0 \leq t \leq T\}$ on the space $(\mathcal{C}_T, \mathcal{B}_T)$ of the continuous functions on $[0,T]$ with the associated Borel σ-algebra $\mathcal{B}_T$ under supremum norm. Let $(\mathcal{C}_T^n, \mathcal{B}_T^n)$ be the subspace of $(\mathcal{C}_T, \mathcal{B}_T)$ generated by $\{X_{t_k}, 0 \leq k \leq n\}$ and $P_\theta^{T,n}$ be the measure P_θ^T restricted to this space. Let Ξ be the Borel σ-algebra of Θ. Then

$$\theta_{n,T} : (\mathcal{C}_T^n, \mathcal{B}_T^n) \to (\Theta, \Xi).$$

Throughout the paper f' denotes derivative w.r.t. θ, f_t denotes derivative w.r.t. t, f_x denotes derivative w.r.t. x of the function f and C denotes a generic constant which may depend on θ but not on anything else.

We assume the following conditions:
(A1) $P_{\theta_1}^{T,n} \neq P_{\theta_2}^{T,n}$ for $\theta_1 \neq \theta_2$ in Θ.

(A2) For each $p > 0$, $\sup_t E|X_t|^p < \infty$.

(A3) $f(\theta, t, x)$ is twice continuously differentiable with respect to θ.

(A4) (i) $|f(\theta, t, x)| \leq L(\theta)(1 + |x|), \theta \in \Theta, x \in \mathbb{R}, t \in [0,T], \sup_{\theta\in\Theta} L(\theta) < \infty$.
(ii) $|f(\theta, t, x) - f(\theta, t, y)| \leq L(\theta)|x - y|, \theta \in \Theta, x, y \in \mathbb{R}, t \in [0,T]$
(iii) $|f(\theta_1, t, x) - f(\theta_2, t, x)| \leq J(x)|\theta_1 - \theta_2|, \theta_1, \theta_2 \in \Theta, x \in \mathbb{R}, t \in [0,T]$
where $J(\cdot)$ is continuous and $\sup_t E[J^2(X_t)] < \infty$.
(iv) $|f'(\theta, t, x) - f'(\theta, t, y)| \leq M(\theta)|x - y|,\ \theta \in \Theta, x, y \in \mathbb{R}, t \in [0,T]$.
(v) f' and f_x satisfy the linear growth condition in x.

(A5) $Q''_{n,T}(\theta)$ is continuous in a neighborhood V_θ of θ for every $\theta \in \Theta$.

(A6) For any $\theta \in \Theta$, there exists a neighborhood V_θ of θ in Θ such that

$$P_\theta \left[\lim_{T\to\infty} \lim_{\frac{T}{n}\to 0} \frac{T}{n} \sum_{k=1}^{n} \{f(\theta_1, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k})\}^2 = \infty \right] = 1$$

for every $\theta_1 \in V_\theta \backslash \{\theta\}$.

(A7) Let $I_{n,T}(\theta) := \sum_{k=1}^{n} f'^2(\theta, t_k, X_{t_k})\Delta t_k$

and $Y_{n,T}(\theta) := \sum_{k=1}^{n} f''^2(\theta, t_k, X_{t_k})\Delta t_k$.

Suppose that there exists non-random function $m_{n,T} \uparrow \infty$ as $T \to \infty$ and $\frac{T}{n} \to 0$ such that

(i) $\frac{I_{n,T}}{m_{n,T}} \stackrel{P_{\theta_0}}{\to} \eta(\theta_0)$ as $T \to \infty$ and $\frac{T}{n} \to 0$

where $P^{T,n}_{\theta_0}(\eta(\theta_0) > 0) > 0$, for all n and T,

(ii) $\frac{Y_{n,T}}{m_{n,T}} \stackrel{P_{\theta_0}}{\to} \xi(\theta_0)$ as $T \to \infty$ and $\frac{T}{n} \to 0$

where $P^{T,n}_{\theta_0}(\xi(\theta_0) > 0) > 0$, for all n, T.

(A8) $P_{\theta_0} \left\{ \int_0^\infty f'^2(\theta_0, t, X_t)dt = \infty \right\} = 1$.

(A9) $P_{\theta_0} - \lim_{T\to\infty} \lim_{\frac{T}{n}\to 0} \frac{I_{n,T}(\theta_0)}{I_T(\theta_0)} = 1$

where $I_T(\theta_0) = \int_0^T f'^2(\theta_0, t, X_t)dt$.

(A10) $E(I_T^{-1}(\theta_0)) \le CT^{-1}$.

3 Strong Consistency

We obtain the strong consistency of the CLSE in this section.

Theorem 3.1 *Let the assumptions (A1) – (A7) hold. Then there exists a root of the normal equation* $Q'_{n,T} = 0$ *which is strongly consistent, i.e.,*

$$\theta_{n,T} \to \theta_0 \quad a.s. \ [P_{\theta_0}] \quad as\ T \to \infty \quad and\ \frac{T}{n} \to 0.$$

Proof. Observe that for any $\delta > 0$,

$$\begin{aligned}
& Q_{n,T}(\theta \pm \delta) - Q_{n,T}(\theta) \\
= & \sum_{k=0}^{n-1} \left[X(t_{k-1}) - X_{t_k} - f(\theta \pm \delta, t_k, X_{t_k}) \frac{T}{n} \right]^2 \\
& - \sum_{k=0}^{n-1} \left[X(t_{k+1}) - X_{t_k} - f(\theta t_k, X_{t_k}) \frac{T}{n} \right]^2 \\
= & \frac{T}{n} \sum_{k=0}^{n-1} \left[f^2(\theta \pm \delta, t_k, X_{t_k}) - f^2(\theta, t_k, X_{t_k}) \right] \\
& -2 \sum_{k=0}^{n-1} \left[f(\theta \pm \delta, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k}) \right] \left[X_{t_{k+1}} - X_{t_k} \right] \\
= & \frac{T}{n} \sum_{k=0}^{n-1} \left[f^2(\theta \pm \delta, t_k, X_{t_k}) - f^2(\theta, t_k, X_{t_k}) \right] \\
& -2 \sum_{k=0}^{n-1} \left[f(\theta \pm \delta, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k}) \right] \left[W_{t_{k+1}} - W_{t_k} \right] \\
& -2 \sum_{k=0}^{n-1} \left[f(\theta \pm \delta, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k}) \right] \int_{t_k}^{t_{k+1}} f(\theta, t, X_t) dt \\
= & \frac{T}{n} \sum_{k=0}^{n-1} \left[f(\theta \pm \delta, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k}) \right]^2 \\
& -2 \sum_{k=0}^{n-1} \left[f(\theta \pm \delta, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k}) \right] \left[W_{t_{k+1}} - W_{t_k} \right] \\
& -2 \sum_{k=0}^{n-1} \left[f(\theta \pm \delta, t_k, X_{t_k}) - f(\theta, t_k, X_{t_k}) \right] \int_{t_k}^{t_{k+1}} \left[f(\theta, t, X_t) - f(\theta, t_k, X_{t_k}) \right] dt \\
=: & M_{1n} + M_{2n} + M_{3n}.
\end{aligned} \tag{3.1}$$

Let us now estimate M_{3n}. For $0 \le k \le n-1$ by assumption (A4), we have

$$\begin{aligned}
& \left| \int_{t_k}^{t_{k+1}} \left[f(\theta, t, X_t) - f(\theta, t_k, X_{t_k}) \right] dt \right| \\
\le \quad & L(\theta) \frac{T}{n} \sup_{t_k \le t \le t_{k+1}} |W_t - W_{t_k}| + L^2(\theta) \frac{T^2}{n^2} \sup_{t_k \le t \le t_{k+1}} \{1 + |X_t|\}.
\end{aligned}$$

Using assumption (A4) again, we obtain

$$M_{3n} \le C(\theta_0) \left\{ \sum_{k=0}^{n-1} \Delta t_k \sup_{t_k \le t \le t_{k+1}} |W_t - W_{t_k}| + \sum_{k=0}^{n-1} \Delta t_k^2 \right\} \delta.$$

Since Θ is compact, it follows that

$$M_{3n} \le C(\theta_0) \left\{ \sum_{k=0}^{n-1} \Delta t_k (2 \Delta t_k \log\log \frac{1}{\Delta t_k})^{1/2} + \sum_{k=0}^{n-1} \Delta t_k^2 \right\} \quad \text{a.s. } [P_{\theta_0}]$$

as $\frac{T}{n} \to 0$ by the law of the iterated logarithm for Brownian motion.

Therefore,

$$M_{3n} = O\left(\frac{T^{3/2}}{n^{1/2}} \log\log^{1/2} \frac{n}{T} \right) \quad \text{a.s. } [P_{\theta_0}]. \tag{3.2}$$

Thus,

$$\begin{aligned}
& Q_{n,T}(\theta_0 \pm \delta) - Q_{n,T}(\theta_0) \\
= & \frac{T}{n} \sum_{k=0}^{n-1} [f(\theta_0 \pm \delta, t_k, X_{t_k}) - f(\theta_0, t_k, X_{t_k})]^2 \\
& -2 \sum_{k=0}^{n-1} [f(\theta_0 \pm \delta, t_k, X_{t_k}) - f(\theta_0, t_k, X_{t_k})] \left[W_{t_{k+1}} - W_{t_k}\right] \\
& +O\left(\frac{T^{3/2}}{n^{1/2}} \log\log^{1/2} \frac{n}{T}\right) \quad \text{a.s.} \quad \left[P_{\theta_0}^{T,n}\right] \\
= & \frac{T}{n} \sum_{k=0}^{n-1} A_k^2(\theta_0) - 2 \sum_{k=0}^{n-1} A_k(\theta_0) \Delta W_k + O\left(\frac{T^{3/2}}{n^{1/2}} \log\log^{1/2} \frac{n}{T}\right) \quad \text{a.s.} \quad \left[P_{\theta_0}^{T,n}\right]
\end{aligned} \tag{3.3}$$

where

$$A_k(\theta_0) := f(\theta_0 \pm \delta, t_k, X_{t_k}) - f(\theta_0, t_k, X_{t_k}). \tag{3.4}$$

Let

$$\frac{T}{n} \sum_{k=0}^{n-1} A_k^2(\theta_0) =: Z_{n,T}. \tag{3.5}$$

Then

$$\begin{aligned}
& \frac{Q_{n,T}(\theta_0 \pm \delta) - Q_{n,T}(\theta_0)}{Z_{n,T}} \\
= & \quad 1 - 2\frac{\sum_{k=0}^{n-1} A_k(\theta_0)\Delta W_k}{Z_{n,T}} + \frac{O\left(\frac{T^{3/2}}{n^{1/2}} \log\log^{1/2} \frac{n}{T}\right)}{Z_{n,T}}.
\end{aligned} \tag{3.6}$$

Since $\sum_{k=0}^{n-1} A_k(\theta_0)\Delta W_k$ is a martingale with respect to the σ-field $\mathcal{F}_T^n$ (the sub σ-field of $\mathcal{F}_T$ generated by $X_{t_k}, 0 < k \le n$) with increasing process $Z_{n,T}$, hence by the SLLN for martingales (see [61, 62])

$$\frac{\sum_{k=0}^{n-1} A_k(\theta_0)\Delta W_k}{Z_{n,T}} \to 0 \quad \text{a.s.} \quad [P_{\theta_0}] \quad \text{as } T \to \infty \quad \text{and} \quad \frac{T}{n} \to 0 \tag{3.7}$$

by assumption (A6).

The last term in the r.h.s. of (3.6) converges to zero a.s. $[P_{\theta_0}]$ as $T \to \infty$ and $\frac{T}{n} \to 0$. Hence from (3.6), we obtain

$$\frac{Q_{n,T}(\theta_0 \pm \delta) - Q_{n,T}(\theta_0)}{Z_{n,T}} \to 1 \quad \text{a.s.} \quad [P_{\theta_0}] \quad \text{as } T \to \infty \quad \text{and} \quad \frac{T}{n} \to 0. \tag{3.8}$$

Furthermore $Z_{n,T} > 0$ a.s. $[P_{\theta_0}]$ by (A1). Therefore, for almost every $w \in \Omega, \delta$ and θ there exist some ϵ, T_0 and n_0 such that for $T \ge T_0$ and $n \ge n_0$ (with $\frac{T_0}{n_0} < \epsilon$)

$$Q_{n,T}(\theta_0 \pm \delta) > Q_{n,T}(\theta_0). \tag{3.9}$$

Since $Q_{n,T}(\theta_0)$ is continuous on the compact set $[\theta_0 - \delta, \theta_0 + \delta]$, it has a local minimum and it is attained at a measurable $\theta_{n,T}$ in $[\theta_0 - \delta, \theta_0 + \delta]$. In view of (3.9), $\theta_{n,T} \in (\theta_0 - \delta, \theta_0 + \delta)$, for $T \ge T_0$ and $n \ge n_0$. Since $Q_{n,T}(\theta)$ is differentiable w.r.t. θ, it follows that $Q'_{n,T}(\theta_{n,T}) = 0$ for $T \ge T_0$ and $n \ge n_0$ such that $\frac{T_0}{n_0} < \epsilon$. Thus $\theta_{n,T} \to \theta_0$ a.s. $[P_{\theta_0}]$ as $T \to \infty$ and $\frac{T}{n} \to 0$. □

4 Asymptotic Normality

We need the following theorem in order to prove the asymptotic normality of $\theta_{n,T}$.

Theorem 4.1 *Under the assumptions (A1) and (A7),*

(a) $\dfrac{Q'_{n,T}(\theta_0)}{I^{1/2}_{n,T}(\theta_0)} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$ *as* $T \to \infty$ *and* $\frac{T}{n^{2/3}} \to 0$.

The convergence is with respect to any measure $\mu \ll P^{\theta_0}_A$ *where* $P^{\theta_0}_A$ *denotes the conditional probability measure given* $A := \{\eta(\theta_0) > 0\}$

(b) $P^{\theta_0}_A - \lim_{T\to\infty} \lim_{\frac{T}{n}\to 0} I_{n,T}(\theta_0) = \infty.$

(c) $P^{\theta_0}_A - \lim_{T\to\infty} \lim_{\frac{T}{n}\to 0} I^{-1}_{n,T}(\theta_0) \sum_{k=0}^{n-1} f''(\theta_0, t_k, X_{t_k})\Delta W_k = 0.$

Proof: First, we will show that

$$I^{-1/2}_{n,T}(\theta_0) \sum_{k=0}^{n-1} f'(\theta_0, t_k, X_{t_k})U_k - I^{-1/2}_T(\theta_0) \int_0^T f'(\theta_0, t, X_t)dW_t \xrightarrow{P^{\theta_0}_A} 0$$

as $T \to \infty$ and $\frac{T}{n^{2/3}} \to 0$, where

$$U_k := X_{t_{k+1}} - X_{t_k} - f(\theta_0, t_k, X_{t_k})\frac{T}{n}.$$

Observe that

$$\begin{aligned}
& I^{-1/2}_{n,T}(\theta_0) \sum_{k=0}^{n-1} f'(\theta_0, t_k, X(t_k))U_k - I^{-1/2}_T(\theta_0) \int_0^T f'(\theta_0, t, X_t)dW_t \\
= \; & \sum_{k=0}^{n-1} f'(\theta_0, t_k, X_{t_k})U_k \left[I^{-1/2}_T(\theta_0) - I^{-1/2}_T(\theta_0)\right] \\
& + I^{-1/2}_T(\theta_0) \left[\sum_{k=0}^{n-1} f'(\theta_0, t_k, X_{t_k})U_k - \int_0^T f'(\theta_0, t, X_t)dW_t\right] \\
= \; & \sum_{k=0}^{n-1} f'(\theta_0, t_k, X_{t_k})U_k \left\{I^{-1/2}_{n,T}(\theta_0) \left[1 - I^{1/2}_{n,T}(\theta_0) I^{-1/2}_T(\theta_0)\right]\right\} \\
& + I^{-1/2}_T(\theta_0) \sum_{k=0}^{n-1} f'(\theta_0, t_k, X_{t_k})(U_k - \Delta W_k) \\
& + I^{-1/2}_T(\theta_0) \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} [f'(\theta_0, t_k, X_{t_k}) - f'(\theta_0, t, X_t)]dW_t \\
=: \; & J_1 + J_2 + J_3.
\end{aligned} \tag{4.1}$$

Let

$$H^{(i)}(t) := f^{(i)}(\theta_0, t_k, X_{t_k}) - f^{(i)}(\theta_0, t, X_t), \quad i = 0, 1, 2, \tag{4.2}$$

whenever $t_k \le t \le t_{k+1}, 0 \le k \le n-1$, where $f^{(i)}$ denotes i-th derivative of f w.r.t. θ.

Now for $0 \le k \le n-1$,

$$E_{\theta_0} \left| I^{-1/2}_T(\theta_0) f'(\theta_0, t_k, X_{t_k})(U_k - \Delta W_k)\right|$$

$$
\begin{aligned}
&= E_{\theta_0}\left\{\left|I_T^{-1/2}(\theta_0)f'(\theta_0,t_k,X_{t_k})\right|\left|\int_{t_k}^{t_{k+1}}[f(\theta_0,t_k,X_{t_k})-f(\theta_0,t,X_t)]dt\right|\right\}\\
&= E_{\theta_0}\left\{\left|I_T^{-1/2}(\theta_0)f'(\theta_0,t_k,X_{t_k})\right|\left|\int_{t_k}^{t_{k+1}}H^{(0)}(t)dt\right|\right\}\\
&\le \left\{E_{\theta_0}\left|I_T^{-1}(\theta_0)f'^2(\theta_0,t_k,X_{t_k})\right|E_{\theta_0}\left|\int_{t_k}^{t_{k+1}}H^{(0)}(t)dt\right|^2\right\}^{1/2} \qquad (4.3)\\
&\le \left\{E_{\theta_0}\left|I_T^{-1}(\theta_0)f'^2(\theta_0,t_k,X_{t_k})\right|\frac{T}{n}\int_{t_k}^{t_{k+1}}E_{\theta_0}[H^{(0)}(t)]^2dt\right\}^{1/2}.
\end{aligned}
$$

But

$$
\begin{aligned}
[H^{(0)}(t)]^2 &= [f(\theta_0,t_k,X_{t_k})-f(\theta_0,t,X_t)]^2\\
&\le 2\left\{|f(\theta_0,t_k,X_{t_k})-f(\theta_0,t_k,X_t)|^2+|f(\theta_0,t_k,X_t)-f(\theta_0,t,X_t)|^2\right\}\\
&\le 2\,L(\theta_0)|X_{t_k}-X_t|^2+2L(\theta_0)|t_k-t|^2 \ \text{ by (A4) (ii).}
\end{aligned}
$$

Hence

$$
\begin{aligned}
E_{\theta_0}[H^{(0)}(t)]^2 &\le 2L(\theta_0)E_{\theta_0}|X_{t_k}-X_t|^2+2L(\theta_0)|t_k-t|^2\\
&\le 2CL(\theta_0)(t_k-t)+2L(\theta_0)(t_k-t)^2\\
&\quad \text{(see [63, p. 48].}
\end{aligned}
$$

Thus

$$
\int_{t_k}^{t_{k+1}}E_{\theta_0}[H^{(0)}(t)]^2dt \le CL(\theta_0)\left(\frac{T}{n}\right)^2+CL(\theta_0)\left(\frac{T}{n}\right)^3
$$

where C denotes a generic constant.

Hence the r.h.s. of (4.3) is

$$
\begin{aligned}
&\le \left\{E_{\theta_0}\left|I_T^{-1}(\theta_0)f'^2(\theta_0,t_k,X_{t_k})\frac{T}{n}\right|\left[CL(\theta_0)\left(\frac{T}{n}\right)^2+CL(\theta_0)\left(\frac{T}{n}\right)^3\right]\right\}^{1/2}\\
&= \left\{E_{\theta_0}\left[\frac{1}{n}\frac{I_{n,T}(\theta_0)}{I_T(\theta_0)}\right]\left[CL(\theta_0)\left(\frac{T}{n}\right)^2+CL(\theta_0)\left(\frac{T}{n}\right)^3\right]\right\}^{1/2}\\
&\le C\frac{T}{n^{3/2}}.
\end{aligned}
$$

First let us estimate $E|J_2|$. Observe that

$$
\begin{aligned}
J_2 &= I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}f'(\theta_0,t_k,X_{t_k})(U_k-\Delta W_k)\\
&= I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}f'(\theta_0,t_k,X_{t_k})\,[f(\theta_0,t,X_t)-f(\theta_0,t_k,X_{t_k})]\,dt
\end{aligned}
$$

By Itô formula, we have

$$
\begin{aligned}
&f(\theta_0,t,X_t)-f(\theta_0,t_k,X_{t_k})\\
&= \int_{t_k}^{t}f_u(\theta_0,u,X_u)du+\frac{1}{2}\int_{t_k}^{t_{k+1}}f_{xx}(\theta_0,u,X_u)du+\int_{t_k}^{t}f_x(\theta_0,u,X_u)dX_u\\
&= \int_{t_k}^{t}[f_u(\theta_0,u,X_u)+f(\theta_0,u,X_u)f_x(\theta_0,u,X_u)+\frac{1}{2}f_{xx}(\theta_0,u,X_u)]du\\
&\quad +\int_{t_k}^{t}f_x(\theta_0,u,X_u)dW_u\\
&=: \int_{t_k}^{t}F(\theta_0,u,X_u)du+\int_{t_k}^{t}f_x(\theta_0,u,X_u)dW_u.
\end{aligned}
$$

Thus

$$\begin{aligned}
E|J_2| &= E\left|I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left[\int_{t_k}^{t} f'(\theta_0,t_k,X_{t_k})f_x(\theta_0,u,X_u)dW_u\right.\right.\\
&\quad\left.\left.+\int_{t_k}^{t} f'(\theta_0,t_k,X_{t_k})F(\theta_0,u,X_u)du\right]dt\right|\\
&\le E\left|I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t} f'(\theta_0,t_k,X_{t_k})f_x(\theta_0,u,X_u)dW_u dt\right|\\
&\quad+E\left|I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t} f'(\theta_0,t_k,X_{t_k})F(\theta_0,u,X_u)dudt\right|\\
&=: D_1+D_2.
\end{aligned}$$

Observe that with $B_{t,k} := \int_{t_k}^{t} f'(\theta_0,t_k,X_{t_k})f_x(\theta_0,u,X_u)dW_u, 0\le k\le n-1$, we have

$$\begin{aligned}
D_1 &= E\left|I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} B_{t,k}dt\right|\\
&\le \left\{E(I_T^{-1}(\theta_0))E(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} B_{t,k}dt)^2\right\}^{1/2}\\
&= \left\{E(I_T^{-1}(\theta_0)\left[\sum_{k=0}^{n-1}E(\int_{t_k}^{t_{k+1}} B_{t,k}dt)^2+\sum_{j\ne k=0}^{n-1}E(\int_{t_k}^{t_{k+1}} B_{t,k}dt)(\int_{t_j}^{t_{j+1}} B_{t,j}dt)\right]\right\}^{1/2}\\
&\le \left\{E(I_T^{-1}(\theta_0))\sum_{k=0}^{n-1}(t_{k+1}-t_k)\int_{t_k}^{t_{k+1}} E(B_{t,k}^2)dt\right\}^{1/2}
\end{aligned}$$

(the last term being zero due to orthogonality of the integrals)

$$\begin{aligned}
&= \left\{E(I_T^{-1}(\theta_0))\sum_{k=0}^{n-1}(t_{k+1}-t_k)\int_{t_k}^{t_{k+1}}\left\{\int_{t_k}^{t} E(f'(\theta_0,t_k,X_{t_k})f_x(\theta_0,u,X_u))^2du\right\}dt\right\}^{1/2}\\
&= \left\{E(I_T^{-1}(\theta_0))\sum_{k=0}^{n-1}(t_{k+1}-t_k)\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\left\{E(f'^2(\theta_0,t_k,X_{t_k})E(f_x^2(\theta_0,u,X_u))\right\}^2 dudt\right\}^{1/2}\\
&\le C\left\{E(I_T^{-1}(\theta_0))\sum_{k=0}^{n-1}(t_{k+1}-t_k)\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\left\{(1+E|X_{t_k}|^2)(1+E|X_u|^2)\right\}^{1/2}dudt\right\}^{1/2}\\
&\quad\text{(by A4(V) and (A2))}\\
&\le E\left\{C(I_T^{-1}(\theta_0))\sum_{k=0}^{n-1}(t_{k+1}-t_k)\int_{t_k}^{t_{k+1}}(t-t_k)dt\right\}^{1/2}\\
&= \left\{CT^{-1}\sum_{k=0}^{n-1}(t_{k+1}-t_k)^3\right\}^{1/2} = C\left(\frac{T^2}{n^2}\right)^{1/2} = C\frac{T}{n}\quad\text{(by (A10)).}
\end{aligned}$$

On the other hand, with $R_{t,k} := \int_{t_k}^{t} f'(\theta_0, t_k, X_{t_k}) F(\theta_0, u, X_u) du, \quad 0 \le k \le n-1$, we have

$$\begin{aligned}
D_2 &= E|I_T^{-1/2}(\theta_0) \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} f'(\theta_0, t_k, X_{t_k}) F(\theta_0, u, X_u) du dt| \\
&\le \left\{ E(I_T^{-1}(\theta_0)) E(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} f'(\theta_0, t_k, X_{t_k}) F(\theta_0, u, X_u) du dt)^2 \right\}^{1/2} \\
&= \left\{ E(I_T^{-1}(\theta_0)) E(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} R_{t,k} dt)^2 \right\}^{1/2} \\
&\le \left\{ E(I_T^{-1}(\theta_0)) \left[\sum_{k=0}^{n-1} (\int_{t_k}^{t_{k+1}} R_{t,k} dt)^2 + \sum_{j \ne k=0}^{n-1} E(\int_{t_k}^{t_{k+1}} R_{t,k} dt)(\int_{t_j}^{t_{j+1}} R_{t,j} dt) \right] \right\}^{1/2} \\
&\le \left[E(I_T^{-1}(\theta_0)) \left\{ (\sum_{k=0}^{n-1} E(\int_{t_k}^{t_{k+1}} R_{t,k} dt)^2 + \sum_{j \ne k=0}^{n-1} \left\{ E(\int_{t_k}^{t_{k+1}} R_{t,k} dt)^2 E(\int_{t_j}^{t_{j+1}} R_{t,j} dt)^2 \right\}^{1/2} \right\} \right]^{1/2} \\
&\le \left[E(I_T^{-1}(\theta_0)) \left\{ \sum_{k=0}^{n-1} (t_{k+1} - t_k) \int_{t_k}^{t_{k+1}} E(R_{t,k}^2) dt \right. \right. \\
&\quad \left. \left. + \sum_{j \ne k=0}^{n-1} \left\{ t_{k+1} - t_k) \int_{t_k}^{t_{k+1}} E(R_{t,k}^2) dt (t_{j+1} - t_j) \int_{t_j}^{t_{j+1}} E(R_{t,j}^2) dt \right\}^{1/2} \right\} \right]^{1/2} \\
&\le \left[E(I_T^{-1}(\theta_0)) \left\{ C \sum_{k=0}^{n-1} (t_{k+1} - t_k)^4 + C \sum_{j \ne k=0}^{n-1} (t_{k+1} - t_k)^2 (t_{j+1} - t_j)^2 \right\} \right]^{1/2} \\
&\le \left\{ C T^{-1} (C n \frac{T^4}{n^4} + \frac{C(n(n-1)}{2} \frac{T^4}{n^4}) \right\}^{1/2} \quad \text{(by (A10))} \\
&\le C(\frac{T^3}{n^2})^{1/2} = C(\frac{T}{n^{2/3}})^{3/2}
\end{aligned}$$

since

$$\begin{aligned}
E(R_{t,k}^2) &= E| \int_{t_k}^{t} f'(\theta_0, t_k, X_{t_k}) F(\theta_0, u, X_u) du|^2 \\
&\le (t - t_k) \int_{t_k}^{t} E|f'(\theta_0, t_k, X_{t_k}) F(\theta_0, u, X_u)|^2 du \\
&\le (t - t_k) \int_{t_k}^{t} \left\{ E(f'^2(\theta_0, t_k, X(t_k)) E(F^2(\theta_0, u, X_u)) \right\}^{1/2} du \\
&\le C(t - t_k) \int_{t_k}^{t} \left\{ (1 + E|X(t_k)|^2)(1 + E|X_u|^2) \right\}^{1/2} du \\
&\qquad \text{(by (A4) and (A2))} \\
&\le C(t - t_k)^2.
\end{aligned}$$

Thus $E|J_2| \to 0$ as $T \to \infty$ and $\frac{T}{n^{2/3}} \to 0$.
Hence

$$J_2 \overset{P_{\theta_0}^{T,n}}{\to} 0 \quad \text{as } T \to \infty \quad \text{and} \quad \frac{T}{n^{2/3}} \to 0. \tag{4.4}$$

From this we obtain

$$\left[\frac{I_{n,T}(\theta_0)}{I_T(\theta_0)}\right]^{1/2} I_{n,T}^{-1/2}(\theta_0)\sum_{k=0}^{n-1} f'(\theta_0,t_k,X_{t_k})(U_k-\Delta W_k)\stackrel{P_{\theta_0}}{\to}0 \text{ as } T\to\infty \text{ and } T/n^{2/3}\to 0$$

which implies

$$\left[\frac{I_{n,T}(\theta_0)}{I_T(\theta_0)}\right]^{1/2} I_{n,T}^{-1/2}(\theta_0)\sum_{k=0}^{n-1} f'(\theta_0,t_k,X_{t_k})U_k\stackrel{P_{\theta_0}}{\to}0 \tag{4.5}$$

$$\text{as } T\to\infty \text{ and } \frac{T}{n^{2/3}}\to 0$$

(by Theorem 4.1(c) and assumption (A9)).

Using (4.5) and assumption (A9) we obtain

$$J_1\stackrel{P_{\theta_0}}{\to}0 \text{ as } T\to\infty \text{ and } \frac{T}{n^{2/3}}\to 0. \tag{4.6}$$

On the other hand,

$$\begin{aligned}
E_{\theta_0}|J_3| &= E_{\theta_0}\left|I_T^{-1/2}(\theta_0)\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} H^{(1)}(t)dW_t\right| \\
&\le \left\{E_{\theta_0}|I_T^{-1}(\theta_0)|E_{\theta_0}\left|\int_0^T H^{(1)}(t)dW_t\right|^2\right\}^{1/2} \\
&= \left\{E_{\theta_0}(I_T^{-1}(\theta_0))\int_0^T E_{\theta_0}[H^{(1)}(t)]^2dt\right\}^{1/2} \\
&= \left\{E_{\theta_0}(I_T^{-1}(\theta_0))\left[nCL(\theta_0)\left(\frac{T}{n}\right)^2+nCL(\theta_0)\left(\frac{T}{n}\right)^3\right]\right\}^{1/2} \\
&\longrightarrow 0 \text{ as } T\to\infty \text{ and } \frac{T}{n}\to 0 \text{ by (A4) and (A8).}
\end{aligned}$$

Hence

$$J_3\stackrel{P_{\theta_0}}{\to}0 \text{ as } T\to\infty \text{ and } \frac{T}{n}\to 0. \tag{4.7}$$

Combination of (4.4), (4.6) and (4.7) proves Theorem 4.1 (a).

(b) follows from (A9) and (A8) since, from (A8) we have $I_\infty(\theta_0)=\infty$ a.s. $[P_{\theta_0}]$.

(c) is a discretized version of the corresponding continuous result in Borkar and Bagchi (1982). We omit the details. In the stationary homogeneous case, the arguments were used by Kasonga (1988).

□

Theorem 4.2 *Under (A1) – (A10),*

$$I_{n,T}^{1/2}(\theta_0)(\theta_{n,T}-\theta_0)\stackrel{\mathcal{D}}{\to}\mathcal{N}(0,1)$$

as $T\to\infty$ and $\frac{T}{n^{2/3}}\to 0$ conditionally. The convergence is w.r.t. any measure $\mu\ll P_A^{\theta_0}$.

Proof. In view of assumption (A5), we can apply Taylor's expansion of $Q'_{n,T}(\theta)$ around $\theta_{n,T}$ and write

$$\begin{aligned}
Q'_{n,T}(\theta_0) &= Q'_{n,T}(\theta_{n,T})+(\theta_0-\theta_{n,T})Q''_{n,T}(\theta_{n,T}+\beta_{n,T}(\theta_0-\theta_{n,T})) \\
&= (\theta_0-\theta_{n,T})Q''_{n,T}(\theta_{n,T}+\beta_{n,T}(\theta_0-\theta_{n,T}))
\end{aligned}$$

where $|\beta_{n,T}|\le 1$ a.s. $[P_{\theta_0}]$ for n and T sufficiently large since $Q'_{n,T}(\theta_{n,T})=0$. Further, since $I_{n,T}>0$ for n and T large with P_{θ_0}- probability approaching one by (A7), hence we have

$$\frac{Q'_{n,T}(\theta_0)}{I_{n,T}^{1/2}(\theta_0)}=\frac{(\theta_0-\theta_{n,T})Q''_{n,T}(\theta_{n,T}+\beta_{n,T}(\theta_0-\theta_{n,T}))}{I_{n,T}^{1/2}(\theta_0)}.$$

Since $\theta_{n,T} \to \theta_0$ a.s. $[P_{\theta_0}]$ as $T \to \infty$ and $\frac{T}{n} \to 0$ by Theorem 3.1 and since $Q''_{n,T}(\theta)$ is continuous by (A5), it follows that

$$Q''_{n,T}(\theta_{n,T} + \beta_{n,T}(\theta_0 - \theta_{n,T})) - Q''_{n,T}(\theta_0) \to 0 \text{ a.s.}[P_{\theta_0}]$$

as $T \to \infty$ and $\frac{T}{n} \to 0$.

Furthermore, $I_{n,T}(\theta_0) \to \infty$ in $P_A^{\theta_0}$-measure as $T \to \infty$ and $\frac{T}{n} \to 0$ by Theorem 4.1 (b). Hence

$$\frac{Q'_{n,T}(\theta_0)}{I^{1/2}_{n,T}(\theta_0)} - \frac{(\theta_0 - \theta_{n,T})Q''_{n,T}(\theta_0}{I^{1/2}_{n,T}(\theta_0)} \stackrel{P_A^{\theta_0}}{\to} 0 \text{ as } T \to \infty \text{ and } \frac{T}{n} \to 0.$$

This property together with Theorem 4.1(a) prove that

$$\frac{(\theta_0 - \theta_{n,T})Q''_{n,T}(\theta_0)}{I^{1/2}_{n,T}(\theta_0)} \stackrel{\mathcal{D}[P_A^{\theta_0}]}{\to} \mathcal{N}(0,1)$$

conditionally as $T \to \infty$ and $\frac{T}{n^{2/3}} \to 0$.

Theorem 4.1(c) shows that

$$\frac{Q'_{n,T}(\theta_0)}{I_{n,T}(\theta_0)} \stackrel{P_A^{\theta_0}}{\to} 1 \text{ as } T \to \infty \text{ and } \frac{T}{n} \to 0.$$

Therefore,

$$I^{1/2}_{n,T}(\theta_0)(\theta_{n,T} - \theta_0) \stackrel{\mathcal{D}[P_A^{\theta_0}]}{\to} \mathcal{N}(0,1) \text{ conditionally as } T \to \infty \text{ and } \frac{T}{n^{2/3}} \to 0.$$

This completes the proof of the theorem. □

5 Examples

(a) Mean Reversion Process with Drift

We verify the conditions of the theorems for the mean reversion model with drift where X_t evolves according to

$$dX_t = \{\theta - \alpha(X_t - \theta t)\}dt + dW_t$$

and $\alpha > 0$ is known. This is a nonhomogeneous generalization of Vasicek model used for modeling short term interest rate in finance, see [57]. The component $-\alpha(X_t - \theta t)$ in the drift function implies that the process is 'forced' to revert to its mean. Here

$$\theta_{n,T} = \frac{\sum_{k=0}^{n-1}(1+\alpha t_k)[X_{t_{k+1}} - X_{t_k}] + \alpha\frac{T}{n}\sum_{k=0}^{n-1}(1+\alpha t_k)X_{t_k}}{\frac{T}{n}\sum_{k=0}^{n-1}(1+\alpha t_k)^2},$$

$$I_{n,T}(\theta_0) = \frac{T}{n}\sum_{k=0}^{n-1}(1+\alpha t_k)^2.$$

We have

$$P_{\theta_0} - \lim_{\frac{T}{n}\to 0} I_{n,T}(\theta_0) = \int_0^T (1+\alpha t)^2 dt,$$

$$P_{\theta_0} - \lim_{T\to\infty}\lim_{\frac{T}{n}\to 0} I_{n,T}(\theta_0) = \lim_{T\to\infty}\int_0^T (1+\alpha t)^2 dt = \infty.$$

Hence (A6) and (A8) hold.

Here

$$m_{n,T} = \frac{T}{n}\sum_{k=1}^{n} t_k^2 e^{\theta g^2(t_k)}$$

hence

$$\frac{I_{n,T}}{m_{n,T}} \stackrel{P_{\theta_0}}{\to} \alpha^2 \quad \text{as } T \to \infty \quad \text{and} \quad \frac{T}{n} \to 0 \quad \text{and} \quad f''(\theta, t, X_t) = 0.$$

Hence condition (A7) is verified. Condition (A9) holds with $I_T(\theta_0) = \int_0^T (1 + \alpha t)dt$. The other conditions are easy to verify. Hence, for the estimator $\theta_{n,T}$, Theorems 3.1, 4.1 and 4.2 hold, i.e., $\theta_{n,T}$ is strongly consistent and conditionally asymptotically normal.

(b) Nonhomogeneous Ornstein-Uhlenbeck process

We verify the conditions of the theorems for the nonhomogeneous Ornstein-Uhlenbeck process $\{X_t\}$ which evolves as the unique solution of the equation

$$dX_t = -\theta g(t) X_t dt + dW_t, \;\; X_0 = 0, \;\; \theta \in \Theta \subset \mathbb{R}\backslash\{0\}$$

where $g : \mathbb{R}^+ \to \mathbb{R}$ is measurable with $\int_0^t g^2(s)ds < \infty$ for every t. In this case

$$\begin{aligned} X_t &= e^{-\theta \int_0^t g(s)ds} \int_0^t e^{\theta \int_0^s g(u)du} dW_u, \\ \theta_{n,T} &= -\frac{\sum_{k=0}^{n-1} g(t_k) X_{t_k} \left(X_{t_{k+1}} - X_{t_k}\right)}{\frac{T}{n}\sum_{k=0}^{n-1} g^2(t_k) X_{t_k}^2} \end{aligned}$$

and

$$I_{n,T} = \frac{T}{n}\sum_{k=0}^{n-1} g^2(t_k) X_{t_k}^2.$$

Let

$$I_T = \int_0^T g^2(t) X_t^2 dt.$$

Note that

$$
\begin{aligned}
& E_{\theta_0}|I_{n,T} - I_T| \\
= \;& E_{\theta_0}\left|\frac{T}{n}\sum_{k=0}^{n-1} g^2(t_k)X_{t_k}^2 - \int_0^T g^2(t)X_t^2 dt\right| \\
= \;& E_{\theta_0}\left|\frac{T}{n}\sum_{k=0}^{n-1} g^2(t_k)X_{t_k}^2 - \sum_{k=0}^{n-1}\int_{t_{k-1}}^{t_k} g^2(t)X_t^2 dt\right| \\
= \;& E_{\theta_0}\left|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^2(t_k)X_{t_k}^2 dt - \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} g^2(t)X_t^2 dt\right| \\
= \;& E_{\theta_0}\left|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left[g^2(t_k)X_{t_k}^2 - g^2(t)X_t^2\right] dt\right| \\
= \;& E_{\theta_0}\left|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left[g(t_k)X_{t_k} - g(t)X_t\right]\left[g(t_k)X_{t_k} + g(t)X_t\right] dt\right| \\
\le \;& \sum_{k=0}^{n-1} E_{\theta_0}\left|\int_{t_k}^{t_{k+1}} \left[g(t_k)X_{t_k} - g(t)X_t\right]^2 dt \int_{t_k}^{t_{k+1}} \left[g(t_k)X_{t_k} + g(t)X_t\right]^2 dt\right|^{\frac{1}{2}} \\
\le \;& \sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}} C(t_k - t)^2 dt \int_{t_k}^{t_{k+1}} L(\theta)\left\{\left(1 + |X_{t_k}|^2\right) + (1 + X_t^2)\right\} dt\right]^{\frac{1}{2}} \\
& \quad \text{(by (A4))} \\
\to \;& 0 \text{ as } T \to \infty \text{ and } \frac{T}{n} \to 0.
\end{aligned}
$$

Hence

$$
P_{\theta_0}^{T,n} - \lim_{\frac{T}{n}\to 0} I_{n,T}(\theta_0) = \int_0^T g^2(t)X_t^2 dt
$$

and

$$
P_{\theta_0}^{T,n} - \lim_{T\to\infty}\lim_{\frac{T}{n}\to 0} I_{n,T}(\theta_0) = \lim_{T\to\infty}\int_0^T g^2(t)X_t^2 t dt = \infty.
$$

Hence (A6) and (A8) hold.

Here

$$
m_{n,T} = \frac{T}{n}\sum_{K=1}^{n} g^2(t_k)e^{\theta g^2(t_k)}.
$$

For simplicity, take $g(t_k) = t_k$.

Then

$$
\begin{aligned}
m_{n,T} &= \frac{T}{n}\sum_{k=0}^{n-1} t_k^2 e^{\theta t_k^2} = \sum_{k=0}^{n-1} k^2(\frac{T}{n})^3 e^{\theta\frac{T^2}{n^2}} \\
&= (\frac{T}{n})^3\frac{n(n+1)(2n+1)}{6}e^{\theta\frac{T^2}{n^2}}, \\
I_{n,T} &= \frac{T}{n}\sum_{k=0}^{n-1} t_k^2 X_{t_k}^2 \\
&= (\frac{T}{n})^3\sum_{k=0}^{n-1} k^2 X_{t_k}^2.
\end{aligned}
$$

Following the method in [64], we have

$$\exp \frac{\theta T^2}{2} X_T \to \zeta \quad \text{a.s. where } \zeta \text{ is } N(0, \left(\frac{\pi}{4\theta}\right)^{1/2})$$

hence it is easy to show that

$$\frac{I_{n,T}}{m_{n,T}} = \frac{6e^{-\theta \frac{T^2}{n^2}} \sum_{k=0}^{n-1} k^2 X^2(t_k)}{n(n+1)(2n+1)} \to \zeta^2 \quad \text{in probability as } T \to \infty \quad \text{and } \frac{T}{n} \to 0.$$

On the other hand

$$f''(\theta, t, X_t) = 0.$$

Hence condition (A7) is verified. Other conditions are easy to verify. Hence the estimator $\theta_{n,T}$ is strongly consistent and asymptotically normally distributed as $T \to \infty$ and $\frac{T}{n^{2/3}} \to 0$.

Remarks

(1) The results obtained here can be generalized to the case of SDE of the type

$$dX_t = f(\theta, t, X_t)dt + g(\theta, t, X_t)dW_t, \quad t \geq 0$$

imposing regularity conditions on the volatility g. Using Doss tranform, this model can be reduced to our model with unit diffusion coefficient.

(2) Berry-Esseen type bounds and large deviation principle for the CLSE remains to be investigated. For Berry-Esseen type bounds of approximate maximum likelihood estimator, see [65].

Competing Interests

Author has declared that no competing interests exist.

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