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The Set of Rationale Numbers is Countably Infinite–A Simple Proof

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Short Research Article

Abstract

This research note presents a very simple proof of the interesting fact that the set Q of rationale numbers is still *countably* infinite as is the set of natural and integer numbers. The proof is based on several innovative ideas and neither relies on Cantor's well-known diagonalization approach nor on the non-trivial Cantor-Schroeder-Bernstein Theorem.

In addition, we present a new proposal for a simple injective function f: Q$\to$Z, which allows one to encode rationals in a highly efficient manner and at the same time it can be understood much more easily (even by non-mathematicians). Moreover, also the inverse function f^{-1} can be derived in an extremely simple manner. Nevertheless, the growth of length is only logarithmic if we compare the resulting length of $f(r=p/q)$ with the value of p, while the length of q has no impact at all on the length of $f(r)$. Our approach also allows us to introduce a total ordering for the set of rationale numbers in a straight-forward manner.

Keywords: *Cardinality of rationals; elementary injective mapping from Q to Z; simplification of Cantor's proof; total-ordering of rationals.*

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1 Proof of Countability of the Set Q

Typically, the fact that the set of rationale numbers has *countably infinite* (for a definition of this term see [1]) elements is proven by means of constructing a bijective mapping between the set Q of rationale numbers and the set N of natural numbers or the set Z of integers, (cf. for example Cantor's method (called: "diagonalization method" or "zig-zag method"), see, e.g., [2] and [3]). An alternate proof of the countability of the set Q can be achieved by defining two injective mappings f: Q→Z as well as a mapping g: Z→Q in combination with the non-trivial Cantor-Schroeder-Bernstein Theorem [4]. By the way, Georg Cantor defined and compared cardinalities as early as in 1878 [5]. Nowadays, mathematicians consider the general topic of "infinite sets" in their research [6].

In this paper we want to present a proof of the countability of Q which is significantly simpler as it requires only one injective mapping, namely f: Q→Z. Our proof makes use of the very helpful fact that by finding an injective mapping f from Q into Z we easily see that f: Q→f(Q) is even a bijection between the sets Q and $f(Q)=\{f(x)| x\in Q\}$. The idea that finding an injective mapping from Q to N (instead of a bijective mapping between Q and Z) for proving that Q is countably infinite has already been published recently [7].

Numerous examples for injective and/or bijective mappings between Q and Z which were proposed up to now, are summarized, e.g., in [8]. However, to the best of our knowledge, all injective mappings$f$: Q→Z proposed until now possess serious weaknesses, like e.g. the injective mapping f, which encodes a ratio r in such a way that the value of the numerator implies a corresponding number of "8"-digits (if r>0) or of "7"-digits (if r<0) followed by "1"-digits according to the value of the denominator. Example: 3/5 is encoded by 88811111. Though, here, encoding and decoding are really straight-forward, we realize easily that the length of the resulting integer number is growing exponentially with the lengths of numerator/ denominator of the ratio being encoded! Despite this quite horrible fact, renowned mathematicians, when this author consulted them (cf. [9]) argued that this mapping function f is considered by them as being the best of the proposals published up to now.

Consulting further mathematicians (all renowned experts in Number/Set Theory) led to the result that, from their point of view, the simplest injective function published up to now is:to encode a ratio $r=p/q$ with $p=p_1p_2\dots p_n$, $q=q_1q_2\dots q_m$ by mapping r onto the integer number $f(r)=(p_1p_2\dots p_nXq_1q_2\dots q_m)_{11}$, where we have to view each digit as being in Base 11.

Example: $r=(3)_{10}/(5)_{10}$ implies $f(r)=(3X5)_{11}$, where X denotes $(10)_{10}$.

Here, the problem of "length explosion" is clearly fixed, but leaving the conventional decimal system and making use of the Base 11 number system (unknown to most non-mathematicians) seems to be quite unacceptable. An additional problem is the considerably high expenditure which is required to achieve both the encoding and the decoding (e.g., finding the ratio r to a given $f(r)$ typically will be very tedious – even a computer could be required for treating large numbers).

This is why we have searched for an injective mapping$f$: Q→Z, which does not possess the disadvantages of the corresponding proposals for f,published up to now. In particular, our new proposal should satisfy the two following requirements:

R1. The output (i.e. the length of $f(r)$) should not become much longer than the input (i.e. the lengths of numerator and of denominator of r).

R2. Both, the encoding function f as well as the decoding (i.e. the inverse function f^{-1}) should be so simple that they can be understood extremely easily – even by non-mathematicians.

Theorem 1. The setQcontains a countably infinite number of elements.

Proof of Theorem 1.

Step 1:

As a first fundamental step of the proof let us construct an *injective* mapping f: Q$\rightarrow$Z, which maps an arbitrary rationale number b in a unique manner onto an integer number $z = f(b) \in$Z. In particular, the function f should satisfy all our requirements R1 and R2 (cf. above). Indeed, we were able to find such a function which will be presented now.

So, let us consider an element $b \in$Q, arbitrarily chosen and then fixed, $b \neq 0$. As b does represent a ratio, we can write b in the following way $b = sgn(b) \bullet \frac{p}{q}$, where $p = p_1 p_2 p_3 \ldots p_n$ denotes the numerator (consisting of n digits p_i, i=1,2,3,…,n) and $q = q_1 q_2 q_3 \ldots q_m$ denotes the denominator (consisting of m digits) and sgn indicates the sign, where $sgn(x) = 0$, for x=0 and $sgn(x) = x/|x|$, for $x \neq 0$.

To achieve a unique encoding of a rationale number (i.e. a ratio) we suppose, without loss of generality, that p and q have no common divisor larger than 1 and are non-negative. First, we assume $b \neq 0$ and we construct f by using an auxiliary number h_b, which starts with exactly n "1"s, $n \geq 1$. So, the number of "1"s indicates the length of the numerator (also n). The "1"s at the beginning of h_b are followed by just one "0", which is then followed by the n digits which constitute the numerator p and these digits are then directly followed by the m digits corresponding to the denominator q.

To summarize, h_b has the following form:

$$h_b = 111\ldots10 p_1 p_2 p_3 \ldots p_n q_1 q_2 q_3 \ldots q_m.$$

Based on our auxiliary number h_b, for arbitrarily chosen elements $b \in$Q, $b \neq 0$, we now can directly obtain the function f we are looking for. The case b=0 can be covered in a straight-forward way by defining $f(0)$=0. So, the complete definition of f is:

$$f(b) = \begin{cases} h_b, & \text{for } b>0. \\ 0, & \text{for } b=0. \\ -h_b, & \text{for } b<0. \end{cases}$$

It is very easy to determine the original number b being mapped on a given number $f(b)$. For example, if $f(b)$=0 we can conclude b=0. In all other cases, we obtain h_b by means of $sgn(f(b)) \bullet f(b) = h_b$, i.e., we obtain h_b easily by just eliminating the sign of $f(b)$. By knowing h_b we know the number of digits of the numerator (indicated by the number of "1"s at the beginning of h_b) and by looking at the n successive digits following the first "0" of h_b we know the complete numerator. The rest of the digits of h_b represent the denominator. The sign of b is identical to the one of $f(b)$.

Therefore, f is a (very simple) injective mapping of Q onto Z, which implies: $|\text{Z}| \geq |\text{Q}|$.

Step 2:

We see that f resp. f^{-1} (the inverse of function f) represent a bijection between Q and f(Q)={$f(x)$ |$x \in$Q}, i.e. f: Q$\rightarrow f$(Q) and f^{-1}: f(Q) $\rightarrow$Q. Therefore, Q and f(Q) possess the same cardinality.

Step 3:

The cardinality of f(Q) is countably infinite because it contains an infinite number of elements and it is a subset of a set containing a countably infinite number of elements (namely the set Z). Therefore, because of the existing bijective mapping between Q and f(Q), it is proven that Q is countably infinite, too.

q.e.d.

It is quite remarkable that our proof of Theorem 1 can easily be generalized to the case that we replace Q by an arbitrary set X possessing an infinite number of elements and the set Z is replaced by a set Y which we assume to possess a countably infinite number of elements. If we still are able to provide an injective mapping f: X $\rightarrow$ Y, an argumentation according to the proof of Theorem1, still proves in this generalized case, too, that X possesses a countably infinite number of elements – a rather general result, indeed.

Lemma 1. Let M be a set possessing a countably infinite number of elements and let M_s be an arbitrary subset of M with an infinite number of elements. Then, M_s also possesses a countably infinite number of elements.

Proof of Lemma 1.

Let the set M possess a countably infinite number of elements. Then, the elements of M are countable (per definitionem). Every (strict) subset of $M_s \subseteq M$ results by eliminating some of the elements of M. Anyway, the elements of M_s still remain countable. Therefore, M_s can only become a finite set or if the set M_s is assumed to possess an infinite number of elements (cf. assumption in Lemma 1) it is proven that M_s is a countably infinite set.

q.e.d.

Remark: The astonishing simplicity of the proof of this Lemma results from the strong assumption underlying Lemma 1, namely the set M is assumed to be countably infinite. If, however, M would be allowed to be uncountably infinite, a subset $M_s \subseteq M$ containing an infinite number of elements, in principle, could be either countably or uncountably infinite. Proving that M_s and M still remain equipotent (having the same cardinality), then will require additional strong assumptions and it will become much more complicated.

Let us now give two examples to illustrate how the original number b can be determined for a given $\boldsymbol{f}(b)$.

Example 1. For $\boldsymbol{f}(b)$ = 11025411 we observe that h_b = 11025411. Thus, the numerator has two digits because h_b has two "1"s in front of the first "0". So, it is easy to see that h_b = 11025411corresponds to ratio b = 25/411.

Boundary between numerator and denominator

Example 2. For $\boldsymbol{f}(b)$ = 1025411 we observe that h_b = 1025411. Thus, the numerator now has only one digit because h_b has only a single "1" in front of the first "0". So, it is easy to see that h_b = 1025411 corresponds to ratio b = 2/5411.

Boundary between numerator and denominator

Some readers may be slightly concerned by the fact that integers resulting from the mapping $f(b)$ of the rational$b = \frac{p}{q}$ onto Z might become quite large numbers (e.g., if n, i.e. the number of digits of p, is rather large). Anyway, there exists a straight-forward solution to eliminate this potential problem. We propose to use the notation $\langle n \rangle p_1 \ldots p_n q_1 \ldots q_m$ to represent the integer $11\ldots10p_1 \ldots p_n q_1 \ldots q_m$ (n times digit "1" at the beginning of this integer) and, therefore, the representation of $f(b)$ becomes much more compact for large values of n. Already for n>2 our proposed notation will reduce the number of symbols required to represent $f(b)$, namely the symbols $\langle$, $\rangle$, and the digits which are used. Encoding now is even significantly less cumbersome than before because the only (trivial) task remaining is to determine n, i.e. the length of p. And, decoding $f(b)$ to determine b is completely trivial now.

Example 3. Using the simplifying, new notation for $b = \frac{123}{1234}$ we obtain $\boldsymbol{f}(b) = \langle 3 \rangle 1231234$.

Remark: Besides, the design of the injective function $\boldsymbol{f}$ represents a nice example for the fact that different scientific disciplines (here, Computer Science and Mathematics) can enrich each other. Due to his long-term experience as a professor of Computer Science focusing on the scientific areas of Data Communication and Computer Networking the author became acquainted with an important approach to efficiently organize the communication between a sender S and a receiver R which exchange signals being interpreted by the receiver as a sequence of ”0”- and ”1”-bits. If no need for data exchange exists the sender just sends ”1”-bits only. In order to structure the communication between S and R, the sender terminates the sequence of ”1”-bits being sent to R and is sending a ”0”-bit. The receipt of a ”0”-bit after a (perhaps long) sequence of ”1”-bits tells R that, directly after receiving the ”0”-bit, receipt of the message to be sent from S to R now starts. Similar to this example from Data Communication the ”0”-digit directly after a sequence of ”1”-digits indicates in the encoding represented by $\boldsymbol{f}$, that the digits of the numerator are beginning after this first ”0”-digit. We see that a principle which is successfully applied in Computer Science can be successfully used in Mathematics, too.

For details regarding the algorithm mentioned to organize the (asynchronous) data transmission between two interconnected computers within a computer network, the interested reader is referred to [10] (cf. asynchronous transmission in Section 6.1, pp. 182-185).

2 Representation of the Set of Rationale Numbers by a Totally-ordered Set

Based on the mapping function $\boldsymbol{f}$, introduced by us, we can obtain a total-ordering of the set of rationale numbers, whose common divisor is not larger than 1. For this purpose, we define an ordering relation **<•** (in words “smaller after being encoded“) which has the property, that either b_A**<•**b_B or b_B**<•**b_A for two arbitrary ratios $b_A \in \mathbf{Q}$ and $b_B \in \mathbf{Q}$, $b_A \neq b_B$. To obtain the solution (regarding the relation **<•**) we compare the integer numbers $\boldsymbol{f}(b_A)$ and $\boldsymbol{f}(b_B)$.

We now define the ordering relation as follows:

$$b_A <\bullet\, b_B \Leftrightarrow \boldsymbol{f}(b_A) < \boldsymbol{f}(b_B)$$
$$b_B <\bullet\, b_A \Leftrightarrow \boldsymbol{f}(b_B) < \boldsymbol{f}(b_A) .$$

Therefore, the relation **<•**implies atotalordering for the set of rationalenumbers for which numerator and denominator do not possess a common divisor larger than 1, because$b_A \neq b_B$ implies that $\boldsymbol{f}(b_A) \neq \boldsymbol{f}(b_B)$ also holds and thus one of both ratios (after being encoded) is smaller than the other.

The assumption that the ratios being compared possess numerator and denominator without a common divisor larger than 1 is required to make sure that there is a unique result when we order both ratios. As an example we choose $b_A = \frac{1}{10}$ and $b_B = \frac{3}{10}$. Then, we get b_A**<•**b_B because 10110 < 10310. However, if we represent b_A by $b_A = \frac{10}{100}$, we suddenly would get b_B**<•**b_A because 10310 < 11010100. Therefore, we assume that $\frac{1}{10}$ for which numerator and denominator do not possess a common divisor larger than 1 does represent the ratio b_A with the unique consequence b_A**<•**b_B.

3 Inversion of the Injective Mapping on to a Subset of the Integers

Unlike a lot of other injective functions $\boldsymbol{f}$: $\mathbf{Q} \rightarrow \mathbf{Z}$, suggested up to now, for our function $\boldsymbol{f}$ it is extremely easy to characterize the elements of $\boldsymbol{f}(\mathbf{Q})$. To demonstrate this shortly, let $\mathbf{Z_0^* := Z^* \cup \{0\}}$, where $\mathbf{Z^*}$ comprises that set of integer numbers which consists of the following numbers $x \in \mathbf{Z}$

$$x = +/-\ x_1 x_2 \ldots x_r x_{r+1} x_{r+2} \ldots x_{2r+1} x_{2r+2} \ldots\ x_s\ ,$$

where$r \geq 1$, $x_k = 1$ for $k \leq r$, $x_{r+1} = 0$, $x_{r+2} \neq 0$, $x_{2r+2} \neq 0$ and $s \geq 2r+2$.

Moreover, we assume that $p = x_{r+2} \ldots x_{2r+1}$ and $q = x_{2r+2} \ldots\ x_s$ do not possess a common divisor larger than 1. We easily see that $\boldsymbol{f}(\mathbf{Q}) = \mathbf{Z_0^*}$. So, $\boldsymbol{f}$ can be used to obtain a bijective mapping between $\mathbf{Q}$ and $\mathbf{Z_0^*}$.

Example 4. We choose x = -11107891234 as an arbitrary element of **Z***. Then x, in a unique manner is mapped onto the rational -789/1234.

Example 5. We now choose an arbitrary ratio, e.g., b = 12/347. Then, in a unique manner,b is mapped onto the integer number$x$ = 11012347. We observe that$x \in$**Z***, because all conditions are fulfilled which are required for the elements of set **Z***, i.e. in particular: $r=2\geq 1$, $x_k=1$ for $k\leq 2$, $x_3=0$, $x_4=1\neq 0$, $x_6=3\neq 0$ and $s=8 \geq 6=2r+2$.

4 Conclusion

A lot of people consider mathematics to represent a rather hard discipline which makes good didactics in Mathematics extremely important [11,12]. Therefore, in particular from a didactic point of view, it seems to be highly desirable to search for proofs of complicated mathematical facts, which are as simple as possible to be understood (as long as they are completely correct). This has been one of the major motivations underlying this paper.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Nykamp DQ.”Countably infinite” definition. From Math Insight.
Available: http://mathinsight.org/definition/countably_infinite

[2] Glasby SP. Enumerating the rationals from left to right. American Mathematical Monthly. 2011;118 (9): 830-835.
DOI: 10.4169/amer.math.monthly.118.09.830

[3] Calkin N, Wilf H. Recounting the rationals. American Mathematical Monthly 2000;107(4): 360-363.
DOI: 10.1080/00029890.2000.12005205

[4] Hinkis A. Proofs of the cantor-bernstein theorem. A mathematical excursion. Science Networks. Historical Studies. Springer. 2013;45.

[5] Cantor G. Ein Beitrag zur Mannigfaltigkeitslehre. Journal für die Reine und Angewandte Mathematik. 1878; (84): 242-248.
DOI: 10.1515/crelle_1878_18788413

[6] Tao T. Infinite sets. In: Analysis I. Texts and readings in mathematics. Springer. Singapore. 2016;37.
Available: https://doi.org/10.1007/978-981-10-1789.6:8

[7] Singh TB. Introduction to Topology. Springer. 2019;422.
ISBN 978-981-13-6954-4

[8] Bradley DM. Counting the Positive Rationals: A Brief Survey. Ann. Appl. Prob. 2005;15:1451-1491.

[9] Private communications between the author and various internationally renowned professors of Mathematics (experts in Number Theory and in Set Theory)

[10] Stallings W. Data and Computer Communications (8th ed.); Prentice-Hall; 2007.

[11] Polya G. How to solve it, a new aspect of mathematical method. Princeton University Press; 2004.

[12] Cummings J. Proofs: A long-form mathematics textbook. (The Long-Form Math Textbook Series); 2021. ISBN-13: 979-8595265 973. Jan. 2021

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