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The Hamiltonian Operator and Euler Polynomials

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper we obtain some identities related to the Hamiltonian operator composed with momentum and position operators and Euler polynomials and confirm these properties through examples.

Keywords: Hamiltonian operator; euler polynomials.

1 INTRODUCTION

Various functions appear in many areas of theoretical physics, for example, Euler polynomials is shown in the field of non-commutative operators in quantum physics. Let us define the commutator of two operators p and q as

$$[p, q] = pq - qp$$

and their anti-commutator as

$$\{p, q\} = pq + qp.$$

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Generally we define the iterated anti-commutators as

$$\{p,q\}_2 = \{\{p,q\},q\},$$
$$\{p,q\}_3 = \{\{\{p,q\},q\},q\} = \{\{p,q\}_2,q\}$$

and moreover for all positive integers n, we have

$$\{p,q\}_n = \{\{p,q\}_{n-1},q\}.$$

We introduce the Hamiltonian operator H as

$$H = \frac{1}{2}\left(p^2 + q^2\right).$$

C. Bender and L. Bettencourt [1] suggest the following result

$$\frac{1}{2^n}\{q,H\}_n = \frac{1}{2}\left\{q, E_n(H+\frac{1}{2})\right\} \tag{1.1}$$

where we can find the Euler polynomials $E_n(x)$ $(n \in \mathbb{N})$ are given by the power series

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2e^{xz}}{e^z+1}. \tag{1.2}$$

The integers $E_n = 2^n E_n(1/2)$ are called Euler numbers. The first few Euler polynomials are

$$\begin{aligned}
E_0(x) &= 1,\\
E_1(x) &= x - \frac{1}{2},\\
E_2(x) &= x^2 - x,\\
E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4},\\
E_4(x) &= x^4 - 2x^3 + x,\\
E_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}.
\end{aligned}$$

It is well-known [2] that

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)x^k \tag{1.3}$$

and

$$E_n(x) + E_n(x+1) = 2x^n \qquad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

In this article we start from the paper [3] and we try to generalize some identities shown on it thus we obtain the following relations of the Hamiltonian operator involving Euler polynomials :

Theorem 1.1. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\\
&\quad\times\left(\left\{q, H - \frac{a}{2}\right\}_k + \left\{q, H - \frac{a}{2} + 1\right\}_k\right)\\
&= \frac{1}{2^{n-1}}\left\{q, H - \frac{a}{2}\right\}_n.
\end{aligned}$$

Corollary 1.2. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then we have*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\\
&\quad\times\left(\left\{q, H - \frac{a}{2}\right\}_k - \left\{q, H - \frac{a}{2} + 2\right\}_k\right)\\
&= \frac{1}{2^{n-1}}\left\{q, H - \frac{a}{2}\right\}_n - \frac{1}{2^{n-1}}\left\{q, H - \frac{a}{2} + 1\right\}_n.
\end{aligned}$$

The interesting thing of these results is that multiplying Hamiltonian operators by Euler polynomials is simply modified to a Hamiltonian operator bracket.

2 SOME IDENTITIES FOR THE HAMILTONIAN OPERATOR

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers and real numbers, respectively. We introduce the symbolic notation, with $a \in \mathbb{R}$,

$$(\{q,H\} + a)_n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}\{q,H\}_k \tag{2.1}$$

and the convention $\{q,H\}_0 = q$.

Proposition 2.1. *(See [3]) For $a \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$\left\{q, H + \frac{a}{2}\right\}_n = \left(\{q,H\} + a\right)_n.$$

Corollary 2.1. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then*

(a)

$$\sum_{k=0}^{n}\binom{2n}{2k}\{q,H\}_{2k}a^{2n-2k} = \frac{1}{2}\left\{q,H+\frac{a}{2}\right\}_{2n}+\frac{1}{2}\left\{q,H-\frac{a}{2}\right\}_{2n},$$

(b)

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}\{q,H\}_{2k+1}a^{2n-2k} = \frac{1}{2}\left\{q,H+\frac{a}{2}\right\}_{2n+1} + \frac{1}{2}\left\{q,H-\frac{a}{2}\right\}_{2n+1}.$$

Proof. By (2.1) and Proposition 2.1 we observe that

$$\left\{q,H+\frac{a}{2}\right\}_n = (\{q,H\}+a)_n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}\{q,H\}_k \tag{2.2}$$

and

$$\left\{q,H-\frac{a}{2}\right\}_n = (\{q,H\}-a)_n = \sum_{k=0}^{n}\binom{n}{k}(-a)^{n-k}\{q,H\}_k. \tag{2.3}$$

(a) After putting $n=2N$ in Eq. (2.2) and (2.3), adding them we obtain

$$2\sum_{k=0}^{N}\binom{2N}{2k}\{q,H\}_{2k}a^{2N-2k} = \sum_{k=0}^{2N}\binom{2N}{k}a^{2N-k}\{q,H\}_k + \sum_{k=0}^{2N}\binom{2N}{k}(-a)^{2N-k}\{q,H\}_k = \left\{q,H+\frac{a}{2}\right\}_{2N}+\left\{q,H-\frac{a}{2}\right\}_{2N}.$$

(b) Let $n=2N+1$ in (2.2) and (2.3). Then adding them we have

$$2\sum_{k=0}^{N}\binom{2N+1}{2k+1}\{q,H\}_{2k+1}a^{2N-2k} = \sum_{k=0}^{2N+1}\binom{2N+1}{k}a^{2N+1-k}\{q,H\}_k + \sum_{k=0}^{2N+1}\binom{2N+1}{k}(-a)^{2N+1-k}\{q,H\}_k = \left\{q,H+\frac{a}{2}\right\}_{2N+1}+\left\{q,H-\frac{a}{2}\right\}_{2N+1}.$$

□

Proposition 2.2. *(See [3]) An equivalent form of identity (1.1) is*

$$\frac{1}{2^n}\left\{q,H-\frac{1}{2}\right\}_n+\frac{1}{2^n}\left\{q,H+\frac{1}{2}\right\}_n = \{q,H^n\}.$$

From the above proposition we consider the following lemma and we can see that Proposition 2.2 is the special case $a=1$.

Lemma 2.2. *Let $n\in\mathbb{N}$ and $a\in\mathbb{R}$. Then we have*

$$\frac{1}{2^n}\left\{q,H-\frac{a}{2}\right\}_n+\frac{1}{2^n}\left\{q,H-\frac{a}{2}+1\right\}_n = \left\{q,\left(H-\frac{a-1}{2}\right)^n\right\}.$$

Proof. From (1.1) we can easily know that

$$\frac{1}{2^n}\left\{q,H-\frac{1}{2}\right\}_n = \frac{1}{2}\{q,E_n(H)\},$$

which deduces that by (1.4)

$$
\begin{aligned}
&\frac{1}{2^n}\left\{q, H-\frac{a}{2}\right\}_n+\frac{1}{2^n}\left\{q, H-\frac{a}{2}+1\right\}_n \\
&=\frac{1}{2}\left\{q, E_n\left(H-\frac{a-1}{2}\right)\right\} \\
&\quad+\frac{1}{2}\left\{q, E_n\left(H-\frac{a-1}{2}+1\right)\right\} \\
&=\left\{q, \frac{1}{2}E_n\left(H-\frac{a-1}{2}\right)\right\} \\
&\quad+\left\{q, \frac{1}{2}E_n\left(H-\frac{a-1}{2}+1\right)\right\} \\
&=\left\{q, \frac{1}{2}\left(E_n\left(H-\frac{a-1}{2}\right)\right.\right. \\
&\quad\left.\left.+E_n\left(H-\frac{a-1}{2}+1\right)\right)\right\} \\
&=\left\{q, \frac{1}{2}\cdot 2\left(H-\frac{a-1}{2}\right)^n\right\} \\
&=\left\{q, \left(H-\frac{a-1}{2}\right)^n\right\}.
\end{aligned}
$$

□

Example 2.3. *In Lemma 2.2 the case $n = 1$ implies that*

$$
\begin{aligned}
&\frac{1}{2}\left(\left\{q, H-\frac{a}{2}\right\}_1+\left\{q, H-\frac{a}{2}+1\right\}_1\right) \\
&=\frac{1}{2}\left(q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)q\right. \\
&\quad\left.+q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)q\right) \\
&=qH+Hq-aq+q \\
&=\left\{q, H-\frac{a-1}{2}\right\}.
\end{aligned}
$$

But since

$$[p, H]=-iq \qquad \text{and} \qquad [q, H]=ip$$

we have

$$
\begin{aligned}
&qH^2-2HqH+H^2q \\
&=[q,H]H-H[q,H] \\
&=[[q,H],H] \\
&=[ip,H] \\
&=i[p,H] \\
&=i(-iq) \\
&=q
\end{aligned}
$$

and

$$HqH=\frac{qH^2+H^2q-q}{2}.$$

This leads for the case $n = 2$ that

$$
\begin{aligned}
&\frac{1}{4}\left(\left\{q, H-\frac{a}{2}\right\}_2+\left\{q, H-\frac{a}{2}+1\right\}_2\right) \\
&=\frac{1}{4}\left(\left\{\left\{q, H-\frac{a}{2}\right\}, H-\frac{a}{2}\right\}\right. \\
&\quad\left.+\left\{\left\{q, H-\frac{a}{2}+1\right\}, H-\frac{a}{2}+1\right\}\right) \\
&=\frac{1}{4}\left(q\left(H-\frac{a}{2}\right)^2+2\left(H-\frac{a}{2}\right)q\left(H-\frac{a}{2}\right)\right. \\
&\quad+\left(H-\frac{a}{2}\right)^2q+q\left(H-\frac{a}{2}+1\right)^2 \\
&\quad+2\left(H-\frac{a}{2}+1\right)q\left(H-\frac{a}{2}+1\right) \\
&\quad\left.+\left(H-\frac{a}{2}+1\right)^2q\right) \\
&=\frac{H^2q}{2}+\frac{qH^2}{2}+HqH-(a-1)Hq \\
&\quad-(a-1)qH+\left(\frac{a^2}{2}-a+1\right)q \\
&=H^2q+qH^2-(a-1)Hq-(a-1)qH \\
&\quad+\left(\frac{a^2}{2}-a+\frac{1}{2}\right)q \\
&=\left\{q, \left(H-\frac{a-1}{2}\right)^2\right\}.
\end{aligned}
$$

Proof of Theorem 1.1. By (2.1) and Proposition 2.1 we note that

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left\{q,H-\frac{a}{2}\right\}_k$$
$$+\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\left\{q,H-\frac{a-2}{2}\right\}_k$$
$$=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\{q,H\}-a\right)_k$$
$$+\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\left(\{q,H\}-a+2\right)_k$$
$$=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\sum_{l=0}^{k}\binom{k}{l}(-a)^{k-l}\{q,H\}_l$$
$$+\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\sum_{l=0}^{k}\binom{k}{l}(-a+2)^{k-l}\{q,H\}_l.$$

Then by replacing $k-l$ with p and using

$$\binom{n}{k}\binom{k}{l}$$
$$=\frac{n!}{k!(n-k)!}\cdot\frac{k!}{l!(k-l)!}$$
$$=\frac{n!}{l!}\cdot\frac{1}{(n-k)!(k-l)!}$$
$$=\frac{n!}{l!(n-l)!}\cdot\frac{(n-l)!}{(n-k)!(k-l)!}$$
$$=\binom{n}{l}\binom{n-l}{k-l}$$
$$=\binom{n}{l}\binom{n-l}{p},$$

(1.3), and (1.4), the above identity becomes

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)$$
$$=\sum_{l=0}^{n}\binom{n}{l}\{q,H\}_l$$
$$\times\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\frac{(-a)^p}{2^{p+l}}$$
$$+\sum_{l=0}^{n}\binom{n}{l}\{q,H\}_l$$
$$\times\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\frac{(-a+2)^p}{2^{p+l}}$$
$$=\sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}$$
$$\times\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\left(\frac{-a}{2}\right)^p$$
$$+\sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}$$
$$\times\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\left(\frac{-a+2}{2}\right)^p$$
$$=\sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}$$
$$\times\left(E_{n-l}(-\frac{a}{2})+E_{n-l}(-\frac{a}{2}+1)\right)$$
$$=\sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}\cdot 2\left(-\frac{a}{2}\right)^{n-l}.$$

This concludes that by (2.1) and Proposition 2.1

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)$$
$$=\frac{1}{2^{n-1}}\sum_{l=0}^{n}\binom{n}{l}\{q,H\}_l(-a)^{n-l}$$
$$=\frac{1}{2^{n-1}}\left(\{q,H\}-a\right)_n$$
$$=\frac{1}{2^{n-1}}\left\{q,H-\frac{a}{2}\right\}_n.$$

□

Example 2.4. *The case $n = 1$ in Theorem 1.1 shows that*

$$\begin{aligned}
&\sum_{k=0}^{1}\binom{1}{k}E_{1-k}(0)\frac{1}{2^k} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_k+\left\{q, H-\frac{a}{2}+1\right\}_k\right) \\
&=\binom{1}{0}E_1(0)\cdot 1\cdot 2q+\binom{1}{1}E_0(0)\cdot\frac{1}{2} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_1+\left\{q, H-\frac{a}{2}+1\right\}_1\right) \\
&=-q+\frac{1}{2}\left(q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)q\right. \\
&\quad\left.+q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)q\right) \\
&=qH+Hq-aq \\
&=\left\{q, H-\frac{a}{2}\right\}_1
\end{aligned}$$

thus it is satisfied. Also if $n = 2$ in Theorem 1.1 then we have

$$\begin{aligned}
&\sum_{k=0}^{2}\binom{2}{k}E_{2-k}(0)\frac{1}{2^k} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_k+\left\{q, H-\frac{a}{2}+1\right\}_k\right) \\
&=\binom{2}{0}E_2(0)\cdot 1\cdot 2q+\binom{2}{1}E_1(0)\cdot\frac{1}{2} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_1+\left\{q, H-\frac{a}{2}+1\right\}_1\right) \\
&\quad+\binom{2}{2}E_1(0)\cdot\frac{1}{4} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_2+\left\{q, H-\frac{a}{2}+1\right\}_2\right) \\
&=\frac{1}{2}\left(\left\{q, H-\frac{a}{2}\right\}_1+\left\{q, H-\frac{a}{2}+1\right\}_1\right) \\
&\quad+\frac{1}{4}\left(\left\{\left\{q, H-\frac{a}{2}\right\}, H-\frac{a}{2}\right\}\right. \\
&\quad\left.+\left\{\left\{q, H-\frac{a}{2}+1\right\}, H-\frac{a}{2}+1\right\}\right) \\
&=-\frac{1}{2}\left(q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)q\right. \\
&\quad\left.+q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)q\right) \\
&\quad+\frac{1}{4}\left(q\left(H-\frac{a}{2}\right)^2+2\left(H-\frac{a}{2}\right)q\left(H-\frac{a}{2}\right)\right. \\
&\quad+\left(H-\frac{a}{2}\right)^2q+q\left(H-\frac{a}{2}+1\right)^2 \\
&\quad+2\left(H-\frac{a}{2}+1\right)q\left(H-\frac{a}{2}+1\right) \\
&\quad\left.+\left(H-\frac{a}{2}+1\right)^2q\right) \\
&=\frac{H^2q}{2}+\frac{qH^2}{2}-aHq-aqH+HqH+\frac{a^2}{2}q \\
&=\frac{1}{2}\left\{q, H-\frac{a}{2}\right\}_2
\end{aligned}$$

and so it is satisfied.

Proof of Corollary 1.2. From Theorem 1.1 we deduce that

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_k-\left\{q, H-\frac{a}{2}+2\right\}_k\right) \\
&=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_k+\left\{q, H-\frac{a}{2}+1\right\}_k\right) \\
&\quad-\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}+1\right\}_k+\left\{q, H-\frac{a}{2}+2\right\}_k\right) \\
&=\frac{1}{2^{n-1}}\left\{q, H-\frac{a}{2}\right\}_n-\frac{1}{2^{n-1}}\left\{q, H-\frac{a}{2}+1\right\}_n.
\end{aligned}$$

□

Example 2.5. *If $n = 1$ in Corollary 1.2 then we obtain*

$$\begin{aligned}
&\sum_{k=0}^{1}\binom{1}{k}E_{1-k}(0)\frac{1}{2^k} \\
&\quad\times\left(\left\{q, H-\frac{a}{2}\right\}_k-\left\{q, H-\frac{a}{2}+2\right\}_k\right) \\
&=-2q \\
&=\left\{q, H-\frac{a}{2}\right\}_1-\frac{1}{2^{n-1}}\left\{q, H-\frac{a}{2}+1\right\}_1.
\end{aligned}$$

And if $n = 2$ in Corollary 1.2 then

$$\sum_{k=0}^{2}\binom{2}{k}E_{2-k}(0)\frac{1}{2^k}$$
$$\times\left(\left\{q,H-\frac{a}{2}\right\}_k-\left\{q,H-\frac{a}{2}+2\right\}_k\right)$$
$$=-2qH-2Hq+2aq-2q$$
$$=\frac{1}{2}\left\{q,H-\frac{a}{2}\right\}_2-\frac{1}{2}\left\{q,H-\frac{a}{2}+1\right\}_2.$$

3 CONCLUSION

We generalized the following identity

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}$$
$$\times\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)$$
$$=\frac{1}{2^{n-1}}\left\{q,H-\frac{a}{2}\right\}_n$$

for $n\in\mathbb{N}$ and $a\in\mathbb{R}$. The case $a=1$ was shown in [3].

COMPETING INTERESTS

Author has declared that no competing interests exist.

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