British Journal of Mathematics & Computer Science
13(3): 1-19, 2016, Article no.BJMCS.22694
ISSN: 2231-0851

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Theory of Approximative Relative Retracts and Its Applications

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Article Information

DOI: 10.9734/BJMCS/2016/22694

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Complete Peer review History: http://sciencedomain.org/review-history/12598

Original Research Article

Received: 20 October 2015
Accepted: 14 November 2015
Published: 08 December 2015

Abstract

In the article a simpler and more generalized perspective of approximative multiretracts is presented (see [1]). This perspective allows for new results and applications. In order to reach it, a class of approximative relative retracts will be defined with the use of single-valued mappings only. Their properties will be studied and some applications to fixed point theory, the theory of the extension of multivalued mappings, to graph-approximation theory and to the theory of approximative retracts will be given.

Keywords: *Relative retract; approximative retract; approximative relative retract; approximative multiretract; multifunction; Vietoris map, finite type; acyclic space; fixed point; approximate selector.*

MSC 2010 Classification: 54C99, 54C55, 55M20.

1 Introduction

In 1953 H. Noguchi introduced the notion of approximative retract (see [2]). In 1970, in [3], J. Jaworowski proved that an approximative retract in the sense of Noguchi is of finite type. Then, in 1971, in [4], M.H. Clapp generalized the notion of approximative retract and proved that such defined space does not have to be of finite type. In 2009, in [1] some more generalized, multivalued version of approximative retract in the sense of Clapp was given. In this article the notion of approximative

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relative retract is introduced. The class of approximative relative retracts is essentially wider than the previous ones and encompasses all thus far known approximative retracts and multiretracts. In this article the properties of approximative relative retracts are studied and it is proven that on some level, and if they are of finite type, they have a fixed point property.

2 Preliminaries

Throughout this paper all topological spaces are assumed to be metrizable. A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. Let X and Y be two spaces and assume that for every $x \in X$ a nonempty and compact subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi : X \multimap Y$ is a multivalued mapping. For a multivalued mapping $\varphi : X \multimap Y$ and a subset $A \subset Y$, we let:

$$\varphi^{-1}(A) = \{x \in X;\ \varphi(x) \subset A\},$$

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then φ is called an upper semi-continuous mapping; we shall write that φ is u.s.c. Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_k(X)\}$ is a graded vector space, $H_k(X)$ being a k-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_{k*}\}$ where $f_{k*} : H_k(X) \to H_k(Y)$ ([5]). A space X is acyclic if:

(i) X is nonempty,

(ii) $H_k(X) = 0$ for every $k \geq 1$ and

(iii) $H_0(X) \approx \mathbb{Q}$.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : u^n(x) = 0 \text{ for some } n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided $dim\widetilde{E} < \infty$.

Let $u = \{u_k\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_k\}$. We call u a Leray endomorphism if

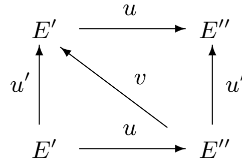
(i) all u_k are admissible,

(ii) almost all $\widetilde{E_k}$ are trivial. For such u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_k (-1)^k tr(\widetilde{u_k}),$$

where $tr(\widetilde{u_k})$ is the ordinary trace of $\widetilde{u_k}$ (comp. [5]). The following important property of the Leray endomorphism is a consequence of the well-known formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace.

Proposition 2.1. *(see [5]) Assume that, in the category of graded vector spaces, the following diagram commutes*



Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case,

$$\Lambda(u') = \Lambda(u'').$$

A proper map $p : X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. A proper map $p : X \to Y$ is called cell-like provided for each $y \in Y$ $p^{-1}(y)$ has a trivial shape in the sense of Borsuk (see [6]). We know that a compact set of trivial shape is acyclic. Hence if $p : X \to Y$ is a cell-like map then it is a Vietoris map. The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y,$$

where $p : Z \to X$ denotes a Vietoris map and $q : Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p, q). We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p : X \to Y$ is a Vietoris map then $p_* : H_*(X) \to H_*(Y)$ is an isomorphism (see [5]).

Definition 2.2.(see [5]) Let $(p, q) \in D(X, Y)$ and $(r, s) \in D(Y, T)$. The composition of the diagrams

$$X \xleftarrow{\ p\ } Z_1 \xrightarrow{\ q\ } Y \xleftarrow{\ r\ } Z_2 \xrightarrow{\ s\ } T,$$

is called the diagram $(u, v) \in D(X, T)$

$$X \xleftarrow{\ u\ } Z_1 \triangle_{qr} Z_2 \xrightarrow{\ v\ } T,$$

$$\text{where } Z_1 \triangle_{qr} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : \ q(z_1) = r(z_2)\},$$

$$u = p \circ f_1, \ v = s \circ f_2,$$

$$Z_1 \xleftarrow{\ f_1\ } Z_1 \triangle_{qr} Z_2 \xrightarrow{\ f_2\ } Z_2,$$

$$f_1(z_1, z_2) = z_1 \text{ (Vietoris map)}, \ f_2(z_1, z_2) = z_2 \text{ for each } (z_1, z_2) \in Z_1 \triangle_{qr} Z_2.$$

It shall be written

$$(u, v) = (r, s) \circ (p, q).$$

In the set of all diagrams $D(X, Y)$, the following relation is introduced:

Definition 2.3. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.

$$(p_1, q_1) \sim_m (p_2, q_2)$$

if and only if there exist spaces Z, Z_1 and Z_2, Vietoris maps $p_3 : Z \to Z_1, p_4 : Z \to Z_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \xleftarrow{\ p_1\ } & Z_1 & \xrightarrow{\ q_1\ } & Y \\ \uparrow{\scriptstyle Id_X} & & \uparrow{\scriptstyle p_3} & & \uparrow{\scriptstyle Id_Y} \\ X & \xleftarrow{\ p\ } & Z & \xrightarrow{\ q\ } & Y \\ \downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle Id_Y} \\ X & \xleftarrow{\ p_2\ } & Z_2 & \xrightarrow{\ q_2\ } & Y, \end{array}$$

that is

$$p = p_1 \circ p_3 = p_2 \circ p_4, \ \ q = q_1 \circ p_3 = q_2 \circ p_4.$$

Proposition 2.4. (see [7]) The relation in the set $D(X, Y)$ introduced in Definition 2.3 is an equivalency relation.

The set of the equivalence classes of the above relation will be denoted by the symbol

$$M_m(X,Y) = D(X,Y)_{/\sim_m}.$$

The elements of the space $M_m(X,Y)$ will be called multimorphisms and will be denoted by

$$\varphi_m = [(p,q)]_m$$

where

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} Y.$$

Proposition 2.5. (see [7]) Let $[(p,q)]_m = \varphi_m \in M_m(X,Y)$.
2.5.1 $((p_1,q_1),(p_2,q_2) \in \varphi_m) \Rightarrow$ (for each $x \in X$ $q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))$),
2.5.2 $((p_1,q_1),(p_2,q_2) \in \varphi_m) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1})$,
2.5.3 Let $\psi_m = [(r,s)]_m \in M_m(Y,T)$ and let $\psi_m \circ \varphi_m = [(r,s)\circ(p,q)]_m \in M_m(X,T)$ (see Definition 2.2). Then for any $(p_1,q_1) \in \varphi_m$ and $(r_1,s_1) \in \psi_m$ we have

$$((r_1,s_1)\circ(p_1,q_1)) \in (\psi_m \circ \varphi_m).$$

From Proposition 2.5 (2.5.1) we get the following definition:

Definition 2.6. For any $\varphi_m \in M_m(X,Y)$, the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_m = [(p,q)]_m$ is called an image of point x in a multimorphism φ_m.

Let $\varphi_m \in M_m(X,Y)$. The symbol $\varphi : X \to_m Y$ will denote a multivalued mapping determined by a multimorphism φ_m (see Definition 2.6). We define (see Proposition 2.5 (2.5.2))

$$\varphi_* = q_* \circ p_*^{-1}, \tag{2.1}$$

where $(p,q) \in \varphi_m$ and if $\psi : Y \to_m T$ then $\psi \circ \varphi : X \to_m T$ is a multivalued map determined by $\psi_m \circ \varphi_m$ (see Proposition 2.5 (2.5.3)) and we have (see [7]):

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*. \tag{2.2}$$

Let $f : X \to Y$ be a continuous map and let $Id_X : X \to X$ be an identity map. Then $f_m = [(Id_X, f)]_m \in M_m(X,Y)$ and for each $(p,q) \in f_m$ (see [7])

$$q_* \circ p_*^{-1} = f_*. \tag{2.3}$$

Let $A \subset X$ be a nonempty set and let $\varphi : X \to_m Y$. Then the map $\varphi_A : A \multimap Y$ given by the formula

$$\varphi_A(x) = \varphi(x) \quad \text{for each } x \in A \tag{2.4}$$

is determined by a multimorphism $(\varphi_A)_m = [(\widetilde{p},\widetilde{q})]_m$ (see [7]), where

$$A \xleftarrow{\;\widetilde{p}\;} p^{-1}(A) \xrightarrow{\;\widetilde{q}\;} Y$$

and $(\widetilde{p},\widetilde{q}) \in D(A,Y)$ is a restriction of some $(p,q) \in \varphi_m$. Hence $\varphi_A : A \to_m Y$ is a multivalued map determined by $(\varphi_A)_m = [(\widetilde{p},\widetilde{q})]_m$.

In this paper, maps determined by multimorphisms $\varphi_m \in M_m(X,Y)$ will be denoted $\varphi : X \to_m Y$ and will be called multifunctions.

Definition 2.7. Let X be an ANR and let $X_0 \subset X$ be a closed subset. We say that X_0 is movable in X provided every neighborhood U of X_0 admits a neighborhood U' of X_0, $U' \subset U$, such that for every neighborhood U'' of X_0, $U'' \subset U$, there exists a homotopy $H\colon U' \times [0,1] \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$.

Definition 2.8. Let X be a compact space. We say that X is movable provided there exists $Z \in ANR$ and an embedding $e : X \to Z$ such that $e(X)$ is movable in Z.

Let us notice that the property of being movable is an absolute property, that is if A is a movable set in some ANR X and $j : A \to X'$ is an embedding into an ANR X', then $j(A)$ is movable in X' (see [6]).

Remark 2.9. [6] We know that movable spaces are of the following types, among others: AR, ANR, $AANR$ (in the sense of Clapp), FAR (of trivial shape) and $FANR$.

Proposition 2.10. [6] Let X and Y be compact spaces. The space $X \times Y$ is movable if and only if X and Y are movable spaces.

Definition 2.11. A map $r \colon Y \to X$ of a space Y onto a space X is said to be an mr-map if there is a multifunction $\varphi : X \to_m Y$ such that $r \circ \varphi = Id_X$.

Definition 2.12. A space X is called an absolute multi-retract (notation: $X \in AMR$) provided there exists a normed space E and an mr-map $r \colon E \to X$ from E onto X.

Definition 2.13. A space X is called an absolute neighborhood multi-retract (notation: $X \in$ANMR) provided there exists an open subset U of some normed space E and an mr-map $r \colon U \to X$ from U onto X.

Class of spaces of type AMR and $ANMR$ are substantially wider than the class AR and ANR respectively (see [8]).

Definition 2.14. Let X be a compact space. We shall say that X is an approximative $ANMR$ (we write $X \in AANMR$) provided that for any $\varepsilon > 0$ there exists a normed space E_ε and an open set $U_\varepsilon \subset E_\varepsilon$, a map $r_\varepsilon : U_\varepsilon \to X$ and a multifunction $\varphi_\varepsilon : X \to_m U_\varepsilon$ such that for any $x \in X$

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon),$$

where $B(x, \varepsilon)$ is an open ball in X with the center of x and radius ε.

Definition 2.15. Let X be a compact space. We shall say that X is an approximative AMR (we write $X \in AAMR$) provided that for any $\varepsilon > 0$ there exists a normed space E_ε, a map $r_\varepsilon : E_\varepsilon \to X$ and a multifunction $\varphi_\varepsilon : X \to_m E_\varepsilon$ such that for any $x \in X$

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon),$$

where $B(x, \varepsilon)$ is an open ball in X with the center of x and radius ε.
In the article [1] the properties and examples of the spaces $AANMR$ and $AAMR$ are presented.

Theorem 2.16. (see [5]) *Let U be an open subset of a normed space E and let X be a compact subset U. Then for every $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a mapping $i_\varepsilon : X \to U$ such that:*

2.16.1 $\|x - i_\varepsilon(x)\| < \varepsilon$ *for all* $x \in X$,

2.16.2 $i_\varepsilon(X) \subset K_\varepsilon$,

2.16.3 i_ε *is homotopic to* i*, where* $i : X \hookrightarrow U$ *is an inclusion.*

We recall that a metrizable space X is of finite type if almost all the homologies of X are trivial and for each $k \geq 0$

$$dim H_k(X) < \infty.$$

Proposition 2.17. Let U be an open subset of a normed space E and let X be a compact subset U. If an inclusion $i : X \hookrightarrow U$ induces a monomorphism $i_* : H_*(X) \to H_*(U)$ then X is of finite type.

Proof. *Let $\varepsilon > 0$ and let $i_\varepsilon : X \to U$ be such as in Theorem 2.16 Let $d : X \to K_\varepsilon$ be a map given by the formula $d(x) = i_\varepsilon(x)$ for each $x \in X$, where K_ε is a finite polyhedron (the condition 2.16.2). We have a following diagram:*

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & U \\ \uparrow{\scriptstyle Id_X} & & \uparrow{\scriptstyle j} \\ X & \xrightarrow{\;d\;} & K_\varepsilon, \end{array}$$

where $i : X \hookrightarrow U$ and $j : K_\varepsilon \hookrightarrow U$ are inclusions. We observe that from the condition 2.16.3 we get

$$i_* = i_{\varepsilon *} = (j \circ d)_* = j_* \circ d_*.$$

From the assumption the map i_* is a monomorphism, so d_* is a monomorphism. Hence X is of finite type and the proof is complete. ☐

A map $\varphi : X \to_m Y$ determined by $\varphi_m = [(p, q)]_m$ is called compact if $q : Z \to Y$ is a compact map ($\overline{q(Z)} \subset Y$ is compact). A map $\varphi : X \to_m X$ has a fixed point (we write $Fix(\varphi) \neq \emptyset$) if there exists a point $x \in X$ such that $x \in \varphi(x)$. We recall that a metrizable space X has a fixed point property (i.e. it is a Lefschetz space) if for each compact multivalued map $\varphi : X \to_m X$ the following condition is satisfied:

$$(\Lambda(\varphi_*) \neq 0) \Rightarrow (Fix(\varphi) \neq \emptyset) \tag{2.5}$$

provided that $\Lambda(\varphi_*)$ (see (2.1)) is well defined.

Propositionn 2.18. [8, 5] Let $X \in ANMR$ (in particular, $X \in ANR$). Then X has a fixed point property.

Propositionn 2.19. [5] Let $g : X \to Y$ be a proper map and let $\varphi_g : Y \multimap X$ be a multivalued map given by the formula $\varphi_g(y) = g^{-1}(y)$ for each $y \in Y$. Then φ_g is an u.s.c. map.

Propositionn 2.20. [5] Let X be a compact set in a Hilbert cube Q. For any open neighborhood U of X in Q there exists a compact space $C \in ANR$ such that $X \subset C \subset U$.

Propositionn 2.21. [9] Let X be a compact and locally connected space and let $f : X \to Y$ be a continuous map from X onto Y. Then Y is compact and locally connected.
Let (X, d_X) be a metric space, $A \subset X$ be a nonempty set and let $\varepsilon > 0$. By the symbol $O_\varepsilon(A)$ will be denoted a following set:

$$O_\varepsilon(A) = \{y \in X;\ \text{there exists } x \in A \text{ such that } d_X(x, y) < \varepsilon\}.$$

Denition 2.22. Let $\varphi : X \multimap Y$ be a multivalued u.s.c. map and let $\varepsilon > 0$. A continuous mapping $f : X \to Y$ is an ε-approximation of φ if and only if for each $x \in X$ $f(x) \in O_\varepsilon(\varphi(O_\varepsilon(x)))$.

Remark 2.23. In this paper we introduce an agreement. Let $\varphi : X \multimap Y$ be a multivalued u.s.c. map. We will say that φ has an approximate selector if for each $\varepsilon > 0$ there exists an ε-approximation of φ.

Let $\mathbb{K}^{n+1}$ be a closed ball in euclidean space $\mathbb{R}^{n+1}$ with the center of 0 and radius 1 and let $\mathbb{S}^n \subset \mathbb{K}^{n+1}$ be a sphere.

Denition 2.24. Let $A \subset X$ be a compact set. We will say that A is ∞-proximally connected subset of X if for every $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that for every $n = 0, 1, 2, ...$ and for every map $g : \mathbb{S}^n \to O_\delta(A)$ there is a map $\widetilde{g} : \mathbb{K}^{n+1} \to O_\varepsilon(A)$ such that $\widetilde{g}(x) = g(x)$ for every $x \in \mathbb{S}^n$.

Remark 2.25. [10, 5] Let $X \subset Q$ be a compact space. Applying Theorem Hyman (see [11]) can be shown that the space X is ∞-proximally connected if and only if it has a trivial shape.

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Theorem 2.26. [5] Let X be a compact ANR and let $\varphi : X \multimap Y$ be an u.s.c. map. Assume that for each $x \in X$ the set $\varphi(x)$ is ∞-proximally connected. Then φ has an approximate selector.

We recall that a space X is of finite type if almost all the homologies of X are trivial and for each $k \geq 0$ $dimH_k(X) < \infty$..

Theorem 2.27. (see [5]) Let X be a compact space of finite type. Then there exists $\varepsilon > 0$ such that for every compact space Y and for every two maps $f, g : Y \to X$ if $d_X(f(y), g(y)) < \varepsilon$ for each $y \in Y$, then $f_* = g_*$, where d_X is a metric in X.

We recall that a compact space X is countably dimensional if

$$X = \bigcup_{n=1}^{\infty} X_n, \text{ where } dimX_n < \infty \text{ for all } n. \tag{2.6}$$

Let $\varphi : X \multimap Y$ be a map and let $A \subset X$ be a nonempty set. We denote $\varphi_A : A \to X$ a map given by the formula $\varphi_A(x) = \varphi(x)$ for each $x \in A$.

Denition 2.28. Let $A \subset X$ $(A \neq X)$ be a nonempty set. Let $\varphi : A \multimap Y$ be an u.s.c. map. An u.s.c. map $\widetilde{\varphi} : X \multimap Y$ will be called an elementary extension of φ if $\widetilde{\varphi}_{X\setminus A} : X\setminus A \to Y$ is a single-valued map and $\widetilde{\varphi}_A = \varphi$.

Proposition 2.29. [12, 13] Let X be a compact space, $Y \in ANR$ $(Y \in AR)$ and let $A \subset X$ $(A \neq X)$ be a nonempty, closed and countably dimensional (in particular, finitely dimensional) set. Assume that $\varphi : A \multimap Y$ is an u.s.c. multivalued map such that for each $x \in X$ the set $\varphi(x)$ is of trivial shape. Then φ has an elementary extension $\widetilde{\varphi} : U \multimap Y$ $(\widetilde{\varphi} : X \multimap Y)$, where $U \subset X$ is some open set such that $A \subset U$.

3 The Relative Retracts

In this section we will give few definitions and we will prove a fact, which we will use in the sections following. The theory of relative retracts can be found in [14]. We denote by Δ a family of compact and nonempty sets in the Hilbert cube Q such that the following conditions are satisfied:

$$\text{for each } x \in Q \quad \{x\} \in \Delta. \tag{3.1}$$

$$\text{if } A_1, A_2, ..., A_n, ... \in \Delta \text{ then } \left(\prod_{n=1}^{\infty} A_n\right) \in \Delta. \tag{3.2}$$

$$\text{if } A \in \Delta \text{ then } h(A) \in \Delta, \text{ where } h : A \to Q \text{ is an embedding.} \tag{3.3}$$

Remark 3.1. The properties of (3.1), (3.2) and (3.3) have, for example, the following family of nonempty sets: the compact sets (see [9], will be write Δ_C), the compact and acyclic sets (see [1], will be write Δ_{CA}), the sets of trivial shape (see [6], will be write Δ_{TS}) and the single element sets (will be write Δ_{SE}).
Let $g : Z \to X$ be a proper map. We will say that g is a Δ map if

$$g^{-1}(x) \in \Delta \text{ for each } x \in X. \tag{3.4}$$

Let $\Re$ denote a set of all metrizable spaces, $X \in \Re$ and let

$$\mathbb{M}(X) = \{f : Z \to X; \; Z \in \Re\}.$$

Let us denote

$$\mathbb{D}(X) = \{g \in \mathbb{M}(X);\ \text{g is a } \Delta \text{ map}\}. \tag{3.5}$$

The examples of families of $\mathbb{D}$ type sets:

$$\mathbb{H}(X) = \{g \in \mathbb{M}(X);\ \text{g is a homeomorphism}\} = \{g \in \mathbb{M}(X);\ \text{g is a } \Delta_{SE} \text{ map}\},$$

$$\mathbb{CE}(X) = \{g \in \mathbb{M}(X);\ \text{g is a cell-like map}\} = \{g \in \mathbb{M}(X);\ \text{g is a } \Delta_{TS} \text{ map}\},$$

$$\mathbb{V}(X) = \{g \in \mathbb{M}(X);\ \text{g is a Vietoris map}\} = \{g \in \mathbb{M}(X);\ \text{g is a } \Delta_{CA} \text{ map}\},$$

$$\mathbb{P}(X) = \{g \in \mathbb{M}(X);\ \text{g is a proper map}\} = \{g \in \mathbb{M}(X);\ \text{g is a } \Delta_{C} \text{ map}\}.$$

We observe that the set $\mathbb{D}(X)$ satisfies the following conditions (see (3.1) and (3.3)):

$$\mathbb{H}(X) \subset \mathbb{D}(X). \tag{3.6}$$

$$(h \in \mathbb{H}(Z) \text{ and } g \in \mathbb{D}(Z, X)) \Rightarrow ((g \circ h) \in \mathbb{D}(X)). \tag{3.7}$$

We observe also that

$$\mathbb{H}(X) \subset \mathbb{CE}(X) \subset \mathbb{V}(X) \subset \mathbb{P}(X). \tag{3.8}$$

Denition 3.2. Let $Z \subset Y$ and let $g : Z \to X$ be a continuous map. A space X is called a g-retract of a space Y (i.e., it is a retract relative to the g) if there exists a continuous map $r : Y \to X$ such that the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{r} & X \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle Id_X} \\ Z & \xrightarrow{g} & X \end{array}$$

is commutative that is $r \circ i = g$, where $i : Z \hookrightarrow Y$ is an inclusion and Id_X is an identity mapping. The space Z will be called a g-carrier of X in Y (we write $Z \in C_Y(X, g)$) and the map r will be called a g-retraction.

Let $\Phi(X)$ be a nonempty subset of $\mathbb{M}(X)$. We will introduce some denotations. Let

$$\Phi(Y, X) = \{g : Y \to X;\ g \in \Phi(X)\},\ \ \Phi_Y(X) = \{g \in \Phi(Z, X);\ Z \subset Y\}.$$

Denition 3.3. A space X is called a $\mathbb{D}_Y(X)$-retract of a space Y (i.e., it is a retract relative to the set $\mathbb{D}_Y(X)$) if there exist maps $g : Z \to X$, $g \in \mathbb{D}_Y(X)$ and $r : Y \to X$ such that r is a g-retraction (see Definition 3.2).

Denition 3.4. We say that a space X is an absolute relative retract (we write $X \in ARR(\mathbb{D})$) if there exists a space Z such that for each space T and for each closed embedding $h : Z \to T$ there exists a map $g \in \mathbb{D}_T(X)$ such that $h(Z) \in C_T(X, g)$ (see Definition 3.2).

Denition 3.5. We say that a space X is an absolute neighborhood relative retract (we write $X \in ANRR(\mathbb{D})$) if there exists a space Z such that for each space T and for each closed embedding $h : Z \to T$ there exists an open set $V \subset T$ and a map $g \in \mathbb{D}_V(X)$ such that $h(Z) \subset V$ and $h(Z) \in C_V(X, g)$ (see Definition 3.2).

It is clear that if X is a compact space and $g : Z \to X$ is a proper map then Z is a compact space. We need the following fact:

Proposition 3.6. Let X be a compact space.
3.6.1 $(X \in ARR(\mathbb{D})) \Leftrightarrow (X$ is a $\mathbb{D}_Q(X)$-retract of the Hilbert cube $Q)$,
3.6.2 $(X \in ANRR(\mathbb{D})) \Leftrightarrow (X$ is a $\mathbb{D}_U(X)$-retract of U, where U is an open set in the Hilbert cube $Q)$.

Proof. We show the condition 3.6.2. *The proof of the condition* 3.6.1 *is analogical. It is obvious that if $X \in ANRR(\mathbb{D})$ then X in particular is a $\mathbb{D}_U(X)$-retract of some open set $U \subset Q$. Assume now that there exists an open set $U \subset Q$ such that X is a $\mathbb{D}_U(X)$-retract of U. Then we get a space $Z \subset U$, a map $g : Z \to X$, $g \in \mathbb{D}_U(X)$ and $r : U \to X$ such that $r \circ i = g$, where $i : Z \hookrightarrow U$ is an inclusion. We have the following diagram:*

$$X \xleftarrow{\;g\;} Z \xrightarrow{\;i\;} U \xrightarrow{\;r\;} X.$$

Let $h : Z \to T$ be a closed embedding, where T is some metrizable space and let $f : h(Z) \to U$ be a map given by the formula $f = i \circ h^{-1}$. $U \in ANR$, so f has a continuous extension $F : V \to U$, where $V \subset T$ is an open set such that $h(Z) \subset V$. Let $R = r \circ F$. Then we have the diagram:

$$X \xleftarrow{\;g\;} Z \xleftarrow{\;h^{-1}\;} h(Z) \xrightarrow{\;j\;} V \xrightarrow{\;R\;} X,$$

where j is an inclusion. We observe that

$$R \circ j = (r \circ F) \circ j = r \circ (F \circ j) = r \circ (i \circ h^{-1}) = (r \circ i) \circ h^{-1} = g \circ h^{-1}.$$

Hence $h(Z) \in C_V(X, g \circ h^{-1})$ (see Definition 3.5), where $(g \circ h^{-1}) \in \mathbb{D}_V(X)$ (see (3.7)) and the proof is complete. ☐

In the paper [14] (see also [8]) we proved the following fact:

Proposition 3.7. Let X be a metrizable space.

3.7.1 $(X \in AMR) \Leftrightarrow (X \in ARR(\mathbb{V}))$,

3.7.2 $(X \in ANMR) \Leftrightarrow (X \in ANRR(\mathbb{V}))$.

4 The Approximative Relative Retracts

We introduce the notion of approximative relative retract and study some of their properties. In this section, all spaces are compact. Let X be a space. By the symbol d_X will be denoted a metric in the space X.

Definition 4.1. Let $\varepsilon > 0$, $Z \subset Y$ and let $g : Z \to X$ be a continuous map. A metrizable space X is called an ε-(g-retract) of a space Y (i.e., it is an ε-retract relative to the g) if there exists a continuous map $r : Y \to X$ such that

$$d_X(r(z), g(z)) < \varepsilon \text{ for each } z \in Z.$$

The space Z will be called an ε-(g-carrier) of X in the space Y (we write $Z \in C^{\varepsilon}_Y(X, g)$) and the map r will be called an ε-(g-retraction).

Definition 4.2. We say that a space X is an approximative absolute relative retract (we write $X \in AARR(\mathbb{D})$) if for each $\varepsilon > 0$ there exists a space Z_ε such that for each space T and for each closed embedding $h : Z_\varepsilon \to T$ there exists $g_\varepsilon : h(Z_\varepsilon) \to X$, $g_\varepsilon \in \mathbb{D}_T(X)$ such that $h(Z_\varepsilon) \in C^{\varepsilon}_T(X, g_\varepsilon)$. Let $\varepsilon > 0$. The space Z_ε will be called an absolute ε-carrier of X and it will write $Z_\varepsilon \in AC^{\varepsilon}(X, \mathbb{D})$.

Definition 4.3. We say that a space X is an approximative absolute neighborhood relative retract (we write $X \in AANRR(\mathbb{D})$) if for each $\varepsilon > 0$ there exists a space Z_ε such that for each space T and for each closed embedding $h : Z_\varepsilon \to T$ there exists $g_\varepsilon : h(Z_\varepsilon) \to X$, $g_\varepsilon \in \mathbb{D}_T(X)$ and an open set $U_\varepsilon \subset T$ such that $h(Z_\varepsilon) \subset U_\varepsilon$ and $h(Z_\varepsilon) \in C^{\varepsilon}_{U_\varepsilon}(X, g_\varepsilon)$. Let $\varepsilon > 0$. The space Z_ε will be called an absolute neighborhood ε-carrier of X and it will write by $Z_\varepsilon \in ANC^{\varepsilon}(X, \mathbb{D})$.

We observe that (see Definition 3.4 and Definition 3.5):

$$(X \in ARR(\mathbb{D})) \Rightarrow (X \in AARR(\mathbb{D})),$$

$$(X \in ANRR(\mathbb{D})) \Rightarrow (X \in AANRR(\mathbb{D})),$$

$$(X \in AARR(\mathbb{D})) \Leftrightarrow (\text{for each } \varepsilon > 0 \;\; AC^{\varepsilon}(X, \mathbb{D}) \neq \emptyset) \text{ and}$$

$$(X \in AANRR(\mathbb{D})) \Leftrightarrow (\text{for each } \varepsilon > 0 \;\; ANC^{\varepsilon}(X, \mathbb{D}) \neq \emptyset).$$

Let Q be a Hilbert cube.

Proposition 4.4. Let X be a metrizable space.
4.4.1 A space $X \in AARR(\mathbb{D})$ if and only if there is a compact space $Z \subset Q$, $g : Z \to X$, $g \in \mathbb{D}_Q(X)$ such that for each $\varepsilon > 0$ $Z \in C^{\varepsilon}_Q(X, g)$.
4.4.2 A space $X \in AANRR(\mathbb{D})$ if and only if there is a compact space $Z \subset Q$, $g : Z \to X$ $g \in \mathbb{D}_Q(X)$ such that for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset Q$ such that $Z \subset U_\varepsilon$ and $Z \in C^{\varepsilon}_{U_\varepsilon}(X, g)$.

Proof. The condition 4.4.2 will be shown. The proof of the condition 4.4.1 is analogical. Let $X \in AANRR(\mathbb{D})$. Then for each n there exists a compact space $Z_n \in ANC^{1/n}(X, \mathbb{D})$. We denote by

$$T = \prod_{n=1}^{\infty} Z_n$$

and let $\pi_n : T \to Z_n$ be a projection. It is clear that T is a compact space. We observe that for each n $Z_n \subset T \subset Q$ (as embedding), so there exists $g_n : Z_n \to X$, $g_n \in \mathbb{D}_Q(X)$, an open set $V_n \subset Q$ such that $Z_n \subset V_n$ and $Z_n \in C^{1/n}_{V_n}(X, g_n)$. We define a space Z given by the formula:

$$Z = \{z \in T;\;\; g_n(\pi_n(z)) = g_{n+1}(\pi_{n+1}(z)) \;\; \textit{for each } n\} = \bigcup_{x \in X} \prod_{n=1}^{\infty} g_n^{-1}(x). \tag{4.1}$$

The space Z is nonempty, compact and $Z \subset T \subset Q$. Let $g : Z \to X$ be a map given by the formula:

$$g(z) = g_1(\pi_1(z)) \;\text{ for each }\; z \in Z.$$

We observe that $g \in \mathbb{D}_Q(X)$ because for each $x \in X$ $g^{-1}(x) = \prod_{n=1}^{\infty} g_n^{-1}(x)$ (see (3.2) and (4.1)) and for each n

$$g(z) = g_n(\pi_n(z)) \;\text{ for each }\; z \in Z. \tag{4.2}$$

Let $\varepsilon > 0$. Then we get n, $r_n : V_n \to X$ such that $1/n < \varepsilon$ and

$$d_X(r_n(y), g_n(y)) < 1/n < \varepsilon \;\text{ for each }\; y \in Z_n. \tag{4.3}$$

The map $f : Z \to V_n$ given by the formula:

$$f(z) = \pi_n(z) \;\text{ for each }\; z \in Z \tag{4.4}$$

has a continuous extension $F_\varepsilon : U_\varepsilon \to V_n$, where $U_\varepsilon \subset Q$ is an open set such that $Z \subset U_\varepsilon$. We show that $Z \in C^{\varepsilon}_{U_\varepsilon}(X, g)$. Let $r_\varepsilon : U_\varepsilon \to X$ be a map given by the formula $r_\varepsilon = r_n \circ F_\varepsilon$ and let $z \in Z$. We have (see (4.2), (4.3) and (4.4))

$$d_X(r_\varepsilon(z), g(z)) = d_X(r_n(F_\varepsilon(z)), g(z)) = d_X(r_n(\pi_n(z)), g_1(\pi_1(z))) =$$

$$= d_X(r_n(\pi_n(z)), g_n(\pi_n(z))) < 1/n < \varepsilon.$$

Assume now that there is a compact space $Z \subset Q$, $g : Z \to X$ $g \in \mathbb{D}_Q(X)$ such that for each $\varepsilon > 0$ there exists an open set $V_\varepsilon \subset Q$ such that $Z \subset V_\varepsilon$ and $Z \in C^{\varepsilon}_{V_\varepsilon}(X, g)$. Let $h : Z \to Y$ be a closed embedding, $\varepsilon > 0$ and let $f_\varepsilon : h(Z) \to Z \subset V_\varepsilon$ be a map given by the formula $f_\varepsilon(x) = h^{-1}(x)$ for each $x \in h(Z)$. There exists a continuous extension $F_\varepsilon : U_\varepsilon \to V_\varepsilon$ of f_ε, where $U_\varepsilon \subset Y$ is an open set such that $h(Z) \subset U_\varepsilon$. We define a map $r_\varepsilon : U_\varepsilon \to X$ by the formula:

$$r_\varepsilon = r'_\varepsilon \circ F_\varepsilon \tag{4.5}$$

where $r'_\varepsilon : V_\varepsilon \to X$ is a map such that for each $z \in Z$

$$d_X(r'_\varepsilon(z), g(z)) < \varepsilon. \tag{4.6}$$

Let $g_\varepsilon = g \circ h^{-1}$ and let $x \in h(Z)$. We have (see (4.5) and (4.6))

$$d_X(r_\varepsilon(x), g_\varepsilon(x)) = d_X(r'_\varepsilon(F_\varepsilon(x)), g(h^{-1}(x))) = d_X(r'_\varepsilon(h^{-1}(x)), g(h^{-1}(x))) < \varepsilon$$

and the proof is complete. □

Analogically to the Proposition 4.4, the following fact can be proven:

Proposition 4.5. Let X be a metrizable space.
4.5.1 A space $X \in AARR(\mathbb{D})$ if and only if there is a normed space E, a compact space $Z \subset E$, $g : Z \to X$, $g \in \mathbb{D}_E(X)$ such that for each $\varepsilon > 0$ $Z \in C^\varepsilon_E(X, g)$.
4.5.2 A space $X \in AANRR(\mathbb{D})$ if and only if there is a normed space E, a compact space $Z \subset E$, $g : Z \to X$ $g \in \mathbb{D}_E(X)$ such that for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset E$ such that $Z \subset U_\varepsilon$ and $Z \in C^\varepsilon_{U_\varepsilon}(X, g)$.
Using the Proposition 4.4 or Proposition 4.5 we introduce the following definition:

Denition 4.6. Let X be a metrizable space.
4.6.1 A space X will be called an approximative absolute neighborhood relative retract in the sense of Noguchi (we write $X \in AANRR_N(\mathbb{D})$) if there exist an open set $U \subset Q$ $(U \subset E)$, a compact space $Z \subset U$, $g : Z \to X$, $g \in \mathbb{D}_Q(X)$ $(g \in \mathbb{D}_E(X))$ such that for each $\varepsilon > 0$ $Z \in C^\varepsilon_U(X, g)$, where E is some normed space.
4.6.2 A space X will be called an approximative absolute neighborhood relative retract in the sense of Clapp (we write $X \in AANRR_C(\mathbb{D})$) if there exist a compact space $Z \subset Q$ $(Z \subset E)$, $g : Z \to X$, $g \in \mathbb{D}_Q(X)$ $(g \in \mathbb{D}_E(X))$ such that for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset Q$ $(U_\varepsilon \subset E)$ such that $Z \subset U_\varepsilon$ and $Z \in C^\varepsilon_{U_\varepsilon}(X, g)$, where E is some normed space.

Definition 4.6 refers only to the spaces of $AANRR(\mathbb{D})$ type because from Proposition 4.4 (the condition 4.4.1) it results that:

$$AARR_C(\mathbb{D}) = AARR_N(\mathbb{D}). \tag{4.7}$$

By $AANR_N$ we denote the class of the approximative absolute neighborhood in the sense of Noguchi and by $AANR_C$ - the class of the approximative absolute neighborhood in the sense of Clapp. We observe that:

$$X \in AARR(\mathbb{H}) \Leftrightarrow X \in AAR,$$

$$X \in AANRR_N(\mathbb{H}) \Leftrightarrow X \in AANR_N,$$

$$X \in AANRR_C(\mathbb{H}) \Leftrightarrow X \in AANR_C.$$

It is clear that if $X \in AANRR_N(\mathbb{D})$ then $X \in AANRR_C(\mathbb{D})$. Later, it will be shown that the inverse implication is not true. We recall that if $g : Z \to X$ is a Vietoris map then $g_* : H_*(Z) \to H_*(X)$ is an isomorphism (see [5]).

Proposition 4.7. (see [3]) Let $X \in AANRR_N(\mathbb{V})$. Then X is of finite type.

Proof. *From Definition 4.6 (the condition 4.6.1) there exist a normed space E, a compact space $Z \subset E$, a map $g : Z \to X$, $g \in \mathbb{V}_E(X)$, an open set $U \subset E$ such that $Z \subset U$ and for each $\varepsilon > 0$ $Z \in C^{\varepsilon}_U(X, g)$. Let for each $\varepsilon > 0$ $r_\varepsilon : U \to X$ be an ε-(g-retraction). Let for each n*

$$Y_n = \{y \in Q : \text{ there exists } x \in X \text{ such that } d_Q(x, y) \leq 1/n\},$$

where d_Q is a metric in Q. We observe that for each n Y_n is compact and

$$X = \bigcap_{n=1}^{\infty} Y_n.$$

From the continuity of Čech homologies it results that a natural homomorphism

$$j_* : H_*(X) \to \lim_{\leftarrow} H_*(Y_n)$$

given by the formula

$$j_*(a) = (j_{1*}(a), ..., j_{n*}(a), ...) \text{ for each } a \in H_*(X) \tag{4.8}$$

is an isomorphism, where for each n j_{n*} is a homomorphism induced by the inclusion $j_n : X \hookrightarrow Y_n$. Let $\varepsilon_n = 1/2n$. We observe that for each $z \in Z$ and for each n the segment $\overline{g(z), r_{\varepsilon_n}(z)}$ lies in Y_n. This implies that for each n there exists a homotopy $h_n : Z \times [0, 1] \to Y_n$ such that

$$h_n(\cdot, 0) = j_n \circ r_{\varepsilon_n} \circ i, \;\; h_n(\cdot, 1) = j_n \circ g,$$

where $i : Z \hookrightarrow U$ is an inclusion. Let $i_*(a) = 0$ for some $a \in H_*(Z)$. Then for each n

$$j_{n*}(g_*(a)) = (j_{n*} \circ g_*)(a) = (j_{n*} \circ r_{\varepsilon_n *} \circ i_*)(a) = (j_{n*} \circ r_{\varepsilon_n *})(i_*(a)) = 0.$$

Hence $g_*(a) = 0$ (see (4.8)) and from the assumption $a = 0$, so the map i_* is a monomorphism. From Proposition 2.17 Z is of finite type. The map g_* is an isomorphism, so X is of finite type and the proof is complete. □

Similarly as in Proposition 4.7, the following fact can be proven (see (4.7)):

Proposition 4.8. Let $X \in AARR(\mathbb{V})$. Then X is an acyclic space.

Proof. Using the Proposition 4.5 (the condition 4.5.1), it is sufficient to adopt $U = E$ in the proof of the Proposition 4.7 . Then the inclusion $i : Z \hookrightarrow E$ induces a monomorphism and, hence, Z is acyclic. The map $g_* : H_*(Z) \to H_*(X)$ is an isomorphism, so X is acyclic and the proof is complete. □

The following fact will be necessary for the construction of an example.

Proposition 4.9. Let X be a metrizable space and let $g : Z \to X$ be a map such that $g \in \mathbb{D}_Q(X)$. Assume that for any n and for each $\varepsilon > 0$ there exist compact spaces $Z_n \subset Z$, $X_n \subset X$ and open neighborhoods U^{ε}_n of Z_n in Q such that

$$Z_n \in C^{\varepsilon}_{U^{\varepsilon}_n}(X_n, g_n) \;\; (Z_n \in C^{\varepsilon}_Q(X_n, g_n)),$$

where $g_n \in \mathbb{D}_Q(X_n)$, $g_n : Z_n \to X_n$. If for any n there exists a continuous map $f_n : Z \to Z_n$ such that $d_X(g_n(f_n(z)), g(z)) < 1/n$ then $X \in AANRR(\mathbb{D})$ $(X \in AARR(\mathbb{D}))$.

Proof. *Let $\varepsilon > 0$. For each n there exists an open set $U_n^{\varepsilon/2} \supset Z_n$ in Q, a map $r_n^{\varepsilon/2} : U_n^{\varepsilon/2} \to X_n$ such that for each $z \in Z_n$*

$$d_X(r_n^{\varepsilon/2}(z), g_n(z)) < \varepsilon/2.$$

Let n be such that $1/n < \varepsilon/2$ and let $F_n^{\varepsilon/2} : V_n^{\varepsilon/2} \to U_n^{\varepsilon/2}$ be an extension of $f_n : Z \to Z_n \subset U_n^{\varepsilon/2}$, where $V_n^{\varepsilon/2}$ is an open set in Q such that $Z \subset V_n^{\varepsilon/2}$. We define a set $V_\varepsilon = V_n^{\varepsilon/2}$ and a map $R_\varepsilon : V_\varepsilon \to X$ by the formula

$$R_\varepsilon = r_n^{\varepsilon/2} \circ F_n^{\varepsilon/2}.$$

Let $z \in Z$, then we have

$$d_X(R_\varepsilon(z), g(z)) < d_X(r_n^{\varepsilon/2}(f_n(z)), g_n(f_n(z))) + d_X(g_n(f_n(z)), g(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and hence $X \in AANRR(\mathbb{D})$. The proof of the second part of the proposition is analogical.

□

Example 4.10. (see [4]) Let

$$l_2 = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} x_n^2 < \infty \right\},$$

where $\mathbb{R}$ is the real numbers set. Let

$$A_n = \left\{ (x_1, x_2, ..., x_{n+1}, 0..., 0, ...) \in l_2 : \left(x_1 - \frac{1}{2n}\right)^2 + x_2^2 + ... + x_{n+1}^2 = \frac{1}{4n^2} \right\}$$

for each $n \geq 1$. Let $a_0 = (0, 0, ..., 0, ...)$. It is clear that $a_0 \in \bigcap_{n=1}^{\infty} A_n$. We define the sets

$$A = \bigcup_{n=1}^{\infty} A_n, \;\; C_n = \bigcup_{m=1}^{n} A_m.$$

We observe that the set A is compact and

$$H_n(B) = \begin{cases} \mathbb{Q} & \text{for } B = A, \\ \{0\} & \text{for } B = (A \backslash A_n) \cup \{a_0\} \end{cases}$$

for each $n \geq 1$. Hence the set A is not of finite type. Let $Y \subset Q$ be a compact and non-movable space such that there exists a cell-like map $p : Q \to Y$ (for each $y \in Y$ the set $p^{-1}(y)$ has a trivial shape, see [15]). We will use Proposition 4.9. Let $X = A \times Y$, $Z = A \times Q$ and let for each n $Z_n = C_n \times Q$, $X_n = C_n \times Y$, $g_n : Z_n \to X_n$ given by the formula $g_n = Id_{C_n} \times p$, where $Id_{C_n} : C_n \to C_n$ is an identity map. We observe that for any n the space $C_n \in ANR$. For any n there exists an open set $U_n \subset Q$ such that $C_n \subset U_n$ and $r_n : U_n \to C_n$ is a retraction. Let for any n $R_n : U_n \times Q \to C_n \times Y$ given by the formula

$$R_n(x, y) = (r_n(x), p(y))$$

for each $(x, y) \in U_n \times Q$ be a g_n-retraction (see Definition 3.2). Then for each $\varepsilon > 0$ and for any n $Z_n \in C_{U_n}^{\varepsilon}(X_n, g_n)$. Let $g : Z \to X$ be a map given by the formula $g = Id_A \times p$. For each n we define a map $f_n : Z \to Z_n$ by the formula:

$$f_n(a, y) = \begin{cases} (a, y) & \text{for } a \in C_n, \\ (a_0, y) & \text{for } a \notin C_n. \end{cases}$$

Hence and from Proposition 4.9 we get that $X \in AANRR_C(\mathbb{V})$ ($X \in AANRR_C(\mathbb{CE})$, because g is a cell-like map). From Proposition 4.7 $X \notin AANRR_N(\mathbb{V})$. From Proposition 2.10 the space X is non-movable, so $X \notin AANR$ (see Remark 2.9).

Example 4.11. We define a set $S \subset \mathbb{R}^2$ by the formula:

$$S = \left(\bigcup_{n=1}^{\infty} \{1/n\} \times [0,1] \right) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$$

and let $T = bd([-1,0] \times [0,1])$ be the boundary of a square. We define the set

$$X = P \times Y,$$

where the space Y is the same as in the Example 4.10 and $P = S \cup T$. We know (see [5, 9]) that X is compact and is not locally connected. The space X is non-movable (see Proposition 2.10). Hence and from Remark 2.9 $X \notin AANR$ (in the sense of Clapp). We show that $X \notin ANRR(\mathbb{V})$ (see Definition 3.5). Assume the contrary that $X \in ANRR(\mathbb{V})$. Then there exist a map $g \in \mathbb{V}_Q(X)$, $g : Z \to X$, an open set $U \subset Q$, a map $r : U \to X$ such that $Z \subset U$ and $r \circ i = g$, where $i : Z \hookrightarrow U$ is an inclusion (see Proposition 3.6). From Proposition 2.20 there exists a compact space $C \in ANR$ such that $X \subset C \subset U$. Let $r_C : C \to X$ be a map given by the formula $r_C(x) = r(x)$ for each $x \in C$. The map r_C is a surjection (g is a surjection), so by Proposition 2.21 the space X must be locally connected, but it is impossible. We show that $X \in AANRR_N(\mathbb{V})$ ($X \in AANRR_N(\mathbb{CE})$). We know that $P \in AANR$ (in the sense of Noguchi, see [2]). There exists an open set $U \subset Q$ such that $P \subset U$ and for each $\varepsilon > 0$ there exists ε-retraction $r_\varepsilon : U \to P$. The map $R_\varepsilon : U \times Q \to P \times Y$ given by the formula

$$R_\varepsilon(x, y) = (r_\varepsilon(x), p(y))$$

is an ε-(g-retraction), where $g : P \times Q \to P \times Y$ is a cell-like map given by the formula $g(x, y) = (x, p(y))$ for each $(x, y) \in P \times Q$ and p is the same as in the Example 4.10.
From the last two examples (see Example 4.10 and Example 4.11), as well as from Proposition 4.4 (see Definition 4.6) it results that

$$\begin{array}{ccccc}
ANRR(\mathbb{D}) & \xrightarrow{\subset} & AANRR_N(\mathbb{D}) & \xrightarrow{\subset} & AANRR_C(\mathbb{D}) \\
\cup\uparrow & & \cup\uparrow & & \cup\uparrow \\
ANR & \xrightarrow{\subset} & AANR_N & \xrightarrow{\subset} & AANR_C,
\end{array}$$

$$\begin{array}{ccccc}
ARR(\mathbb{D}) & \xrightarrow{\subset} & AARR_N(\mathbb{D}) & \xrightarrow{=} & AARR_C(\mathbb{D}) \\
\cup\uparrow & & \cup\uparrow & & \cup\uparrow \\
AR & \xrightarrow{\subset} & AAR_N & \xrightarrow{=} & AAR_C
\end{array}$$

and none of the inclusions can be reversed. Finally it will be proven that the old definition of the approximative absolute multi-retract (Definition 2.14 and Definition 2.15, see [1]) overlaps with the definition of the approximative absolute relative retract (Definition 4.2, Definition 4.3, Proposition 4.4 and Proposition 3.7).

Proposition 4.12. Let X be a metrizable space.
4.12.1 $X \in AAMR \Leftrightarrow X \in AARR(\mathbb{V})$,
4.12.2 $X \in AANMR \Leftrightarrow X \in AANRR(\mathbb{V})$.

Proof. The condition 4.12.2 will be shown. The proof of the condition 4.12.1 is analogical. Let $X \in AANMR$. Then for any $\varepsilon > 0$ there exists a normed space E_ε and an open set $U_\varepsilon \subset E_\varepsilon$, a map $r_\varepsilon : U_\varepsilon \to X$ and a multifunction $\varphi_\varepsilon : X \to_m U_\varepsilon$ such that for any $x \in X$

$$r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon),$$

where φ_ε is determined by $(\varphi_\varepsilon)_m = [(p_\varepsilon, q_\varepsilon)]_m$ and

$$X \xleftarrow{p_\varepsilon} Z_\varepsilon \xrightarrow{q_\varepsilon} U_\varepsilon.$$

From Lemma 5.3 (see below) for each $\varepsilon > 0$ and for each $z \in Z_\varepsilon$ we get

$$d_X(r_\varepsilon(q_\varepsilon(z)), p_\varepsilon(z)) < \varepsilon.$$

Let $\varepsilon > 0$. Let $h_\varepsilon : Z_\varepsilon \to T$ be an embedding and let $F_\varepsilon : V_\varepsilon \to U_\varepsilon$ be an extension of $q_\varepsilon \circ h_\varepsilon^{-1}$, where V_ε is an open set in T such that $h_\varepsilon(Z_\varepsilon) \subset V_\varepsilon$. We have

$$X \xleftarrow{p_\varepsilon} Z_\varepsilon \xleftarrow{h_\varepsilon^{-1}} h_\varepsilon(Z_\varepsilon) \xrightarrow{i_\varepsilon} V_\varepsilon \xrightarrow{F_\varepsilon} U_\varepsilon \xrightarrow{r_\varepsilon} X,$$

where i_ε is an inclusion. Assume that $r'_\varepsilon = r_\varepsilon \circ F_\varepsilon$, $p'_\varepsilon = p_\varepsilon \circ h_\varepsilon^{-1}$ and $Z'_\varepsilon = h_\varepsilon(Z_\varepsilon)$. Let $z \in Z'_\varepsilon$. Then we have

$$d_X(r'_\varepsilon(z), p'_\varepsilon(z)) = d_X(r_\varepsilon(F_\varepsilon(z)), p_\varepsilon(h_\varepsilon^{-1}(z))) = d_X(r_\varepsilon(q_\varepsilon(h_\varepsilon^{-1}(z))), p_\varepsilon(h_\varepsilon^{-1}(z))) < \varepsilon.$$

It is clear that $p'_\varepsilon \in \mathbb{V}_T(X)$. Hence $X \in AANRR(\mathbb{V})$. Assume now that $X \in AANRR(\mathbb{V})$. Then from Proposition 4.5 (the condition 4.5.2) there is a normed space E, compact space $Z \subset E$, $g : Z \to X$ $g \in \mathbb{V}_E(X)$ such that for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset E$ such that $Z \subset U_\varepsilon$ and $Z \in C^\varepsilon_{U_\varepsilon}(X, g)$. For each $\varepsilon > 0$ there exists a continuous map $r_\varepsilon : U_\varepsilon \to X$ such that $d_X(r_\varepsilon(z), g(z)) < \varepsilon$, where

$$X \xleftarrow{g} Z \xrightarrow{i_\varepsilon} U_\varepsilon \xrightarrow{r_\varepsilon} X.$$

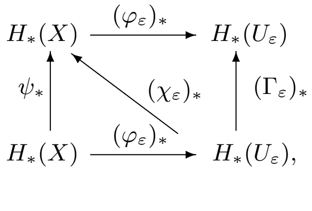
Let $q_\varepsilon = i_\varepsilon$, $p_\varepsilon = g$ and let φ_ε be determined by $(\varphi_\varepsilon)_m = [(p_\varepsilon, q_\varepsilon)]_m$. Then from Lemma 5.3 (see below) the proof is complete. □

5 The Applications

In this section some of the applications of approximative relative retracts will be given. Let us assume that all spaces are compact. With the use of methods known in mathematical literature, it will be proven that under some assumption approximative relative retracts have a fixed point property (see (2.5)).

Proposition 5.1. *Let $X \in AANRR_C(\mathbb{V})$. If X is of finite type then X has a fixed point property.*

Proof. From the assumption there exists a compact space $Z \subset Q$, a Vietoris map $p : Z \to X$ such that for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset Q$ such that $Z \subset U_\varepsilon$ and $r_\varepsilon : U_\varepsilon \to X$ is a ε-(p-retraction) (see Proposition 4.4). Let $\psi : X \to_m X$ be a multifunction determined by $\psi_m = [(p', q)]_m$ and let for each $\varepsilon > 0$ $\varphi_\varepsilon : X \to_m U_\varepsilon$ be a multifunction determined by $(\varphi_\varepsilon)_m = [(p, i_\varepsilon)]_m$, where $i_\varepsilon : Z \hookrightarrow U_\varepsilon$ is an inclusion. We observe that for each $\varepsilon > 0$ $\varphi_\varepsilon(X) \subset Z$. We have the following diagrams:



where $\chi_\varepsilon = \psi \circ r_\varepsilon$ and $\Gamma_\varepsilon = \varphi_\varepsilon \circ \chi_\varepsilon$. From Theorem 2.27 there exists an $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ the diagram is commutative. Indeed, we have

$$(\chi_\varepsilon)_* \circ (\varphi_\varepsilon)_* = (\psi \circ r_\varepsilon)_* \circ (\varphi_\varepsilon)_* = (\psi \circ r_\varepsilon)_* \circ (i_\varepsilon \circ p^{-1})_* = \psi_* \circ ((r_\varepsilon)_* \circ (i_\varepsilon)_*) \circ p_*^{-1} =$$

$$= \psi_* \circ (r_\varepsilon \circ i_\varepsilon)_* \circ p_*^{-1} = \psi_* \circ (p_* \circ p_*^{-1}) = \psi_*,$$

because $(r_\varepsilon \circ i_\varepsilon)_* = p_*$ and it is clear that $(\Gamma_\varepsilon)_* = (\varphi_\varepsilon)_* \circ (\chi_\varepsilon)_*$. Hence and from Proposition 2.1 $\Lambda(\psi_*)$ is well defined (for each $\varepsilon > 0$ $U_\varepsilon \in ANR$ and Γ_ε is a compact multifunction) and $\Lambda(\psi_*) = \Lambda((\Gamma_\varepsilon)_*)$. Assume now that $\Lambda(\psi_*) \neq 0$. Then for each $\varepsilon \in (0, \varepsilon_1]$ $\Lambda((\Gamma_\varepsilon)_*) \neq 0$ and there exists a fixed point $y_\varepsilon \in \Gamma_\varepsilon(y_\varepsilon) \subset Z$ (see Proposition 2.18). Hence

$$p(y_\varepsilon) \in p(\Gamma_\varepsilon(y_\varepsilon)) = p(\varphi_\varepsilon(\psi(r_\varepsilon(y_\varepsilon)))) = \psi(r_\varepsilon(y_\varepsilon)).$$

For each $\varepsilon \in (0, \varepsilon_1]$ $d_X(p(y_\varepsilon), r_\varepsilon(y_\varepsilon)) < \varepsilon$, so ψ has an ε-fixed point. The space X is compact and ψ is an u.s.c, so ψ has a fixed point and the proof is complete. □

Now a few facts, necessary to formulate a very important examples, will be proven. Let X be a metrizable space, $x \in X$ and $v > 0$. By the symbol $B(x, v)$ will be denoted an open ball in X with the center of x and radius v.

Proposition 5.2. *Let X be a metrizable space. Assume that for each $\varepsilon > 0$ there exist an open set $U_\varepsilon \subset Q$, an u.s.c map $\varphi_\varepsilon : X \multimap U_\varepsilon$ and a continuous map $r_\varepsilon : U_\varepsilon \to X$ such that the following conditions are satisfied:*

5.2.1 *for each $x \in X$ $r_\varepsilon(\varphi_\varepsilon(x)) \subset B(x, \varepsilon)$,*

5.2.2 *φ_ε has an approximate selector. Then $X \in AANR$. If for each $\varepsilon > 0$ $U_\varepsilon = Q$ then $X \in AAR$.*

Proof. Let $\varepsilon > 0$. Then for $\varepsilon/3$ there exist $\varphi_{\varepsilon/3} : X \multimap U_{\varepsilon/3}$, $r_{\varepsilon/3} : U_{\varepsilon/3} \to X$ such that

$$r_{\varepsilon/3}(\varphi_{\varepsilon/3}(x)) \subset B(x, \varepsilon/3) \text{ for each } x \in X. \tag{5.1}$$

Let $V \subset Q$ be an open set such that $\varphi_{\varepsilon/3}(X) \subset V$ and $\overline{V} \subset U_{\varepsilon/3}$. The map $r_{\varepsilon/3}$ is uniformly continuous on a compact set $\overline{V}$. Hence there exists $\delta > 0$ such that for each $y, z \in \overline{V}$ we have:

$$(d_Q(y, z) < \delta) \Rightarrow (d_X(r_{\varepsilon/3}(y), r_{\varepsilon/3}(z)) < \varepsilon/3), \tag{5.2}$$

where d_X is a metric in X and d_Q is a metric in Q. We can assume that $\delta < \varepsilon/3$ and $O_\delta(\varphi_{\varepsilon/3}(X)) \subset V$. From 5.2.2 and Definition 2.22, we get the continuous map $f : X \to U_{\varepsilon/3}$ such that for each $x \in X$

$$f(x) \in O_\delta(\varphi_{\varepsilon/3}(O_\delta(x))).$$

Let $x \in X$. Then there exist $s \in O_\delta(x)$ and $z \in \varphi_{\varepsilon/3}(s)$ such that

$$d_Q(f(x), z) < \delta < \varepsilon/3. \tag{5.3}$$

From (5.1), (5.2) and (5.3) we have

$$d_X(r_{\varepsilon/3}(f(x)), x) \leq d_X(r_{\varepsilon/3}(f(x)), r_{\varepsilon/3}(z)) + d_X(r_{\varepsilon/3}(z), s) + d_X(s, x) < \varepsilon.$$

Let $F_{\varepsilon/3} : V_{\varepsilon/3} \to U_{\varepsilon/3}$ be a continuous extension of f, where $V_{\varepsilon/3} \subset Q$ is an open set such that $X \subset V_{\varepsilon/3}$. Let $V'_\varepsilon = V_{\varepsilon/3}$ and let $s_\varepsilon : V'_\varepsilon \to X$ be a map given by the formula

$$s_\varepsilon = r_{\varepsilon/3} \circ F_{\varepsilon/3}.$$

Then the map s_ε is an ε-retraction. The proof of the second part of the proposition is now obvious. □

Lemma 5.3. Let $\varepsilon > 0$. Let $\varphi : X \multimap Y$ be a map given by the formula $\varphi(x) = f(g^{-1}(x))$ for each $x \in X$ and let $r : Y \to X$ be a map, where $g \in \mathbb{D}(X)$ and

$$X \xleftarrow{\;g\;} Z \xrightarrow{\;f\;} Y \xrightarrow{\;r\;} X.$$

$$(r(\varphi(x)) \subset B(x,\varepsilon) \text{ for each } x \in X) \Leftrightarrow (d_X(r(f(z)), g(z)) < \varepsilon \text{ for each } z \in Z).$$

Proof. Assume that $r(\varphi(x)) \subset B(x,\varepsilon)$ for each $x \in X$. Let $z \in Z$. Then there exists $x \in X$ such that $z \in g^{-1}(x)$. Hence $r(f(z)) \in B(x,\varepsilon)$ and we have:

$$d_X(r(f(z)), g(z)) = d_X(r(f(z)), x) < \varepsilon.$$

Assume now that $d_X(r(f(z)), g(z)) < \varepsilon$ for each $z \in Z$. Let $x \in X$ and let $z \in g^{-1}(x)$. Then we get:

$$d_X(r(f(z)), x) = d_X(r(f(z)), g(z)) < \varepsilon.$$

Hence $r(\varphi(x)) \subset B(x,\varepsilon)$ and the proof is complete. □

From Proposition 5.2 and Lemma 5.3 we get the following fact:

Proposition 5.4. Let $X \in AANRR(\mathbb{D})$ ($X \in AARR(\mathbb{D})$). Let $g : Z \to X$ be a map such that $g \in \mathbb{D}_Q(X)$ and for any $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset Q$ such that $Z \subset U_\varepsilon$ and $Z \in C^{\varepsilon}_{U_\varepsilon}(X, g)$ ($Z \in C^{\varepsilon}_{Q}(X, g)$) (see Proposition 4.4). Let $i_\varepsilon : Z \hookrightarrow U_\varepsilon$ be an inclusion. If for each $\varepsilon > 0$ the multifunction $\varphi_\varepsilon : X \to_m U_\varepsilon$ ($\varphi_\varepsilon : X \to_m Q$) determined by $(\varphi_\varepsilon)_m = [(g, i_\varepsilon)]_m$ has an approximate selector then $X \in AANR$ ($X \in AAR$).

Approximative relative retracts can be used for characterization of approximative retracts in the countably dimensional and compact spaces.

Proposition 5.5. Let $X \in AANRR(\mathbb{CE})$ ($X \in AARR(\mathbb{CE})$). Assume that X is countably dimensional (in particular, finitely dimensional). Then $X \in AANR$ ($X \in AAR$).

Proof. Let $X \in AANRR(\mathbb{CE})$. Then there exists a compact space $Z \subset Q$, $g \in \mathbb{CE}_Q(X)$, $g : Z \to X$ such that for each $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset Q$ such that $Z \subset U_\varepsilon$ and $Z \in C^{\varepsilon}_{U_\varepsilon}(X, g)$. We take a positive number ε. Let $i_\varepsilon : Z \hookrightarrow U_\varepsilon$ be an inclusion and let $\varphi_\varepsilon : X \to_m U_\varepsilon$ be a multifunction determined by $(\varphi_\varepsilon)_m = [(g, i_\varepsilon)]_m$. From Proposition 2.29 there exists an elementary extension $\widetilde{\varphi_\varepsilon} : V \to_m U_\varepsilon$ of φ_ε, where $V \subset Q$ is an open set such that $X \subset V$. There exists a compact $C \in ANR$ such that $X \subset C \subset V$ (see Proposition 2.20). By Theorem 2.26 (see Remark 2.25) the map $\widetilde{(\varphi_\varepsilon)}_C : C \to_m U_\varepsilon$ has an approximate selector. Hence φ_ε has an approximate selector and by Proposition 5.4 $X \in AANR$. The proof of the second part of the proposition is analogical. □

It is clear that $AANR_N \subset AANRR_N(\mathbb{CE})$ and $AANR_C \subset AANRR_C(\mathbb{CE})$. These inclusions cannot be reversed (see Examples 4.10 and 4.11). With the use of Proposition 5.4, the aforementioned examples will be presented. We observe that the assumption $X \in ANR$ in Theorem 2.26 is necessary.

Example 5.6. Let X be a compact and non-movable space. From Remark 2.9 it result that $X \notin ANR$. It can be shown that $X \in ARR(\mathbb{V})$ (see [8, 14]). Let $p : Q \to X$ be a map such that for each $x \in X$ $p^{-1}(x)$ has a trivial shape (see [15]). We define a multifunction $\varphi : X \to_m Q$ by the formula $\varphi(x) = p^{-1}(x)$ for each $x \in X$. The map φ is determined by $\varphi_m = [(p, Id_Q)]_m$ and for each $x \in X$ the set $\varphi(x)$ is of trivial shape (it is ∞-proximally connected, see Remark 2.25). We will show that φ does not have an approximate selector. Assume the contrary that there exists an approximate selector of φ. By adopting in the Proposition 5.4 for each $\varepsilon > 0$ $\varphi_\varepsilon = \varphi$, $U_\varepsilon = Q$ and $r_\varepsilon = p$ (see Proposition 5.2) we get that $X \in AAR$, but it is a contradiction, because X is a non-movable space (see Remark 2.9).

The construction of Example 5.6 is possible due to some properties of approximative relative retracts. The next example, in turn, illustrates the application of approximative relative retracts to the theory of the extension of multivalued mappings.

Example 5.7. Let $\varphi : X \to_m Q$ be a multifunction such as in Example 5.6. Assume that there exists an open set $V \subset Q$ such that $X \subset V$ and there exists an extension $\widetilde{\varphi} : V \to_m Q$ such that for each $x \in V$ the set $\widetilde{\varphi}(x)$ is of trivial shape (it is ∞-proximally connected, see Remark 2.25). Then from Proposition 2.20 there exists a compact space $C \in ANR$ such that $X \subset C \subset V$. By Theorem 2.26 the map $\widetilde{\varphi}_C : C \to_m Q$ given by the formula $\widetilde{\varphi}_C(x) = \widetilde{\varphi}(x)$ for each $x \in C$ has an approximate selector. Hence φ has an approximate selector, but as we know (see Example 5.6), it is impossible. This means that φ cannot be extended to any open set $V \subset Q$ such that $X \subset V$ although the $Q \in AR$.

6 Conclusions

From Example 5.6 two conclusions can be drawn. Firstly, the assumption in Theorem 2.26 that $X \in ANR$ is essential and cannot be omitted. The mapping $\varphi : X \to_m Q$ in Example 5.6 does not satisfy one assumption of Theorem 2.26, i.e. $X \notin ANR$ ($X \in ARR(\mathbb{V})$, see [8, 14]) and does not have an approximate selector. Secondly, Example 5.7 contains a response to Suszycki's question from [16] (p. 187). The mapping $\varphi : X \multimap Q$ does not have an extension to a mapping of the same images onto any open neigborhood of space X. To conclude, in the article three levels of approximative retracts can be distinguished. The first level consists of approximative retracts of $AANRR(\mathbb{H})$ type that are approximative retracts of $AANR$ type. The second level is an essentially wider class of approximative retracts of $AANRR(\mathbb{V})$ type that shows similar properties to the $AANR$ class and encompassing the class of earlier defined $AANMR$. The third level, in turn, is a very generalized level of approximative retracts of $AANRR(\mathbb{D})$ type is founded on the $\mathbb{D}$ type set. At this level the fact that allowed for the division of the approximative relative retracts into two essentially different classes was proven. Moreover, it was proven that the classes $AANRR_N(\mathbb{D})$, $AANRR_C(\mathbb{D})$ are essentially wider than the classes, respectively, $AANR_N$, $AANR_C$.

Competing Interests

The author declares that no competing interests exist.

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